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Holographic Action for the Self-Dual Field

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Abstract

We revisit the construction of self-dual field theory in $4\ell + 2$ dimensions using Chern-Simons theory in $4\ell + 3$ dimensions, building on the work of Witten. Careful quantization of the Chern-Simons theory reveals all the topological subtleties associated with the self-dual partition function, including the generalization of the choice of spin structure needed to define the theory. We write the partition function for arbitrary torsion background charge, and in the presence of sources. We show how this approach leads to the formulation of an action principle for the self-dual field.

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1 Introduction

The problem of formulating a quantum self-dual field is an important part of the formulation of string theory and supergravity. It is very subtle. It was pointed out some time ago by Marcus and Schwarz [1] that there is no simple Poincaré invariant action principle for the self-dual gauge field. Since then, much has been written about the action and the quantization of the self-dual field. For an incomplete sampling of the literature see [2]-[26]. Nevertheless, we believe the last word has not yet been said on this problem. The main point of the present paper is to describe a new approach to the formulation of an action principle for self-dual fields. Our main motivation for writing the paper is that we needed to solve this problem more thoroughly than heretofore before writing a corresponding action for RR fields in type IIA and IIB supergravity. The action for RR fields will be described in a separate publication.

The action principle is described below in the introduction and in section 7. In a word, the action is a period matrix defined by a canonical complex structure and a *choice* of Lagrangian decomposition of fieldspace. We arrive at this principle following the lead of Witten [27, 28], who stressed that the best way to formulate a self-dual theory is to rely on a Chern-Simons theory in one higher dimension. This is the “holographic approach” of the title. One advantage of this approach is that it takes proper account of topological aspects ignored in other discussions. These are not — as is often stated — minor topological subtleties, but can lead to qualitative physical effects. Even in the simplest example of a chiral scalar in $1 + 1$ dimensions, that chiral scalar is equivalent to a chiral Weyl fermion. Accordingly, one cannot formulate the theory without making a choice of spin structure. We will explain how the spin structure is generalized and how the theory depends upon it. The holographic approach has other advantages: It is the correct way to capture the subtle half-integer shifts in Dirac quantization laws for the fieldstrength. It is also a good way to approach the question of the metric dependence of the self-dual partition function — a subject of some relevance to stabilization of string theory moduli. Not much is known about this metric dependence, and we take some initial steps towards understanding it.

Our route to the action proceeds by careful construction of the self-dual partition function. The self-dual partition function has been already discussed by Witten in [27, 28] and by Hopkins and Singer in [29] (see also [30, 31, 32, 33]). Nevertheless, there were some technical points in these papers which we found confusing and we hope our work will add some useful clarification. In particular, we hope that our paper will make clear the physical relevance of the main theorem of Hopkins and Singer in [29]. The basic principle we employ for writing the action has in fact already been used in [34], but the discussion of [34] was restricted to the harmonic sector of the fields. Here we have generalized it to the full infinite-dimensional fieldspace and broadened the

context in a way we believe will be useful.

1.1 Main result

Let us now describe our results in more technical detail. Consider a $4\ell + 2$ -dimensional space-time manifold M equipped with a Lorentzian metric of signature $- + \cdots +$. The Hodge $*$ squares to $+1$ on the middle dimensional forms $\Omega^{2\ell+1}(M)$, making it possible to impose a self-duality constraint on a field strength $\mathcal{F} \in \Omega^{2\ell+1}(M)$:

$$*_g \mathcal{F}^+ = \mathcal{F}^+. \quad (1.1)$$

When we impose (1.1) the Bianchi identity and equation of motion coincide

$$d\mathcal{F}^+ = 0. \quad (1.2)$$

A classical field theory describing the self-dual particle is completely specified by these two equations. The quantum theory, however, is problematic. As we have noted, folklore states there is no straightforward Lorentz covariant action. Moreover, an important aspect of the quantum theory is Dirac quantization. In the string theory literature many authors attempt to impose a Dirac quantization condition of the form

$$\mathcal{F}^+ \in \Omega_{\mathbb{Z}}^{2\ell+1}(M), \quad (1.3)$$

i.e. \mathcal{F}^+ is a closed form with integral periods. However this quantization condition is *incompatible* with the self-duality constraint (1.1) since the self-duality condition (1.1) varies continuously with the metric g . As we will see, both of these difficulties are nicely overcome by the holographic approach.

For technical reasons (e.g. use of the Hodge theorem) it is much more convenient to work on a manifold with Riemannian metric. Let (X, g_E) be a compact Riemannian $4\ell + 2$ -dimensional manifold. Now the Hodge $*_E$ squares to -1 on the space of $2\ell + 1$ forms. So the self-dual form becomes *imaginary anti-self-dual*:

$$*_E R^+ = -iR^+. \quad (1.4)$$

In section 2 below we will explain the key insight that the partition function of an imaginary anti self-dual field on a $4\ell + 2$ -manifold X should be viewed as a *wave function* of an abelian spin Chern-Simons theory on a $4\ell + 3$ -manifold Y where X is a component of ∂Y . The Chern-Simons field plays the role of a current coupling to the self-dual field. The wavefunction as a function of the Chern-Simons field $A \in \Omega^{2\ell+1}(X)$ is the self-dual partition function as a function of an external current.

The following heuristic argument should make this connection to Chern-Simons theory quite plausible [35]. The Chern-Simons action $\text{CS} \sim \int_Y A \wedge dA$, so on a manifold with boundary

$\delta\text{CS} = \int_X \delta A \wedge A$. To get a well-posed boundary value problem we set $A = *A|_X$. But in Chern-Simons theory the gauge modes in the bulk, Y , become dynamical fields (“edge states”) on the boundary X . In this case the gauge freedom is $A \rightarrow A + R$, $R \in \Omega_{\mathbb{Z}}^{2\ell+1}(Y)$, so we have a dynamical field with $R = *R$ on the boundary.

When we go beyond this heuristic level we find that the Chern-Simons theory depends on an integer level k .¹ Even defining the Chern-Simons term at the fundamental level $k = 1$ turns out to be very subtle indeed, and this leads to the most difficult aspects in the work of Hopkins, Singer and Witten. In the case of the self-dual scalar in two dimensions the corresponding Chern-Simons theory in three dimensions is at “half-integer level,” (this corresponds to $k = 1$ in our normalization) and is known as spin Chern-Simons theory. In this case, in addition to a level, one must also specify a spin structure even to define the Chern-Simons term. In the higher dimensional case there is an analogous choice generalizing the choice of spin structure. We must stress the word “generalizing”; in physical applications we do *not* want to restrict attention to spin manifolds for $\ell > 0$. Since the term “generalized spin structure” is already in use for something entirely different [36], we will refer to our generalized spin structure as a *QRIF* — for reasons to be explained below.

Of course, the difficulties in defining the $k = 1$ Chern-Simons term only arise in the presence of topologically nontrivial fields. In section 3 we describe how to formulate fieldspace in a way that properly accounts for topology. The space of gauge equivalence classes of fields is formulated in terms of differential cohomology. A trickier aspect is how to describe gauge potentials, and here we take a somewhat pragmatic approach. At the cost of some mathematical naturality, we gain in physical insight. We then review some aspects of the Hopkins-Singer theory in section 4, but an understanding of this theory is not strictly necessary in order to follow the rest of the paper: we will make an end-run around their key theorem, to be described presently.

In section 5 we turn to the real technical work, the quantization of the Chern-Simons theory. Once we have understood how to formulate the action, the quantization of this free topological theory follows the standard pattern. The physical space of states is the space of wavefunctions satisfying the Gauss law. For level $k = 1$ it turns out that the Chern-Simons theory has a 1-dimensional Hilbert space. As we vary the external current and the metric, both of which couple to the self-dual field, the partition function thus becomes a covariantly constant section of a line bundle with connection. Therefore, up to a constant, the construction of the partition function is thus the construction of this line bundle with connection.


¹A slight generalization, described in section 2, shows that there is in fact a theory for any pair of integers p, q . The level is then given by $k = pq$. In another kind of generalization, one can assume that \mathcal{F} takes values in a vector space equipped with an involution $I^2 = \pm 1$. Such generalizations naturally arise in compactifications of self-dual theories. This is related to the generalization where k can be taken to be an integral matrix. A thorough study in the three-dimensional case of the latter generalization can be found in [37].

Partition function. Let us sketch briefly the construction of the partition function. (A much more precise discussion is the subject of sections 5 and 6). It is important for the whole construction that the space $V_{\mathbb{R}} = \Omega^{2\ell+1}(X, \mathbb{R})$ is a real symplectic vector space with the symplectic form


$$\omega(u, v) = \int_X u \wedge v. \quad (1.5)$$

To get a wave function we need to choose a polarization on the phase space. This can be obtained by choosing the Hodge complex structure $J = -*_E$ on $V_{\mathbb{R}}$. Using this complex structure we decompose the space of forms as

$$V_{\mathbb{R}} \otimes \mathbb{C} \cong V^+ \oplus V^-.$$



imaginary anti self-dual



imaginary self-dual

To any real vector $R \in V_{\mathbb{R}}$ we associate R^{\pm} in the complex vector space V^{\pm} by

$$R^{\pm} = \frac{1}{2}(R \pm i *_E R). \quad (1.6)$$

We may now quantize using holomorphic polarization. Holomorphy and gauge invariance fix the dependence on the Chern-Simons field to be essentially a “theta function” in infinite dimensions. The precise formula is given in Theorem 6.1 and Corollary 6.1. When restricted to the harmonic fields the wavefunction is essentially a theta function on the finite-dimensional torus $\mathcal{H}^{2\ell+1}(X)/\mathcal{H}_{\mathbb{Z}}^{2\ell+1}(X)$ where $\mathcal{H}^{2\ell+1}(X)$ is the space of harmonic $(2\ell+1)$ -forms and $\mathcal{H}_{\mathbb{Z}}^{2\ell+1}(X)$ is the lattice of harmonic forms with integral periods. Denoting by a the Chern-Simons field, the partition function in the harmonic sector takes the form

$$\mathcal{Z} \left[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon_2 \end{smallmatrix} \right] (a^+, a^-) = \mathcal{N}_g e^{i\pi \int_X a^- \wedge a^+} \vartheta \left[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon_2 \end{smallmatrix} \right] (a^+ | \tau) \quad (1.7)$$

where $\vartheta \left[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon_2 \end{smallmatrix} \right]$ is a theta function with characteristics ε_1 and ε_2 (see Theorem 6.2 or appendix B for a definition), \mathcal{N}_g is a normalization factor which captures the metric dependence, and τ is a complex period matrix. It is completely determined by the metric and a choice of Lagrangian decomposition of the lattice of harmonic forms with integer periods: $\bar{\Gamma}_1^h \oplus \bar{\Gamma}_2^h = \mathcal{H}_{\mathbb{Z}}^{2\ell+1}(X)$. Equation (1.7) is only a caricature. See Theorem 6.2 for the precise result. In particular it hides some important subtleties to which we will soon return. But before that, in the next two subsections, we will reconsider the two main problems with the quantum theory mentioned above in light of (1.7).

1.2 Action and classical equation of motion

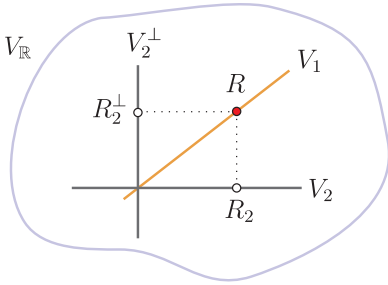
In section 7 we describe an action principle for the self-dual field. We give a brief summary of that action here. The relation of the partition function to a theta function suggests the proper way to

formulate the action. For simplicity we put $a = 0$ and $\varepsilon = 0$ in (1.7). From the definition of the theta function as an infinite sum we learn that the period matrix can be viewed as the on-shell action in the harmonic sector of the theory:

$$S_E(R) = i\pi\tau(R^+) \quad (1.8)$$

where $R^+ = \frac{1}{2}(R + i *_E R)$ and $R \in \bar{\Gamma}_1^h$.² Now we need to extend equation (1.8) to the vector space $V_{\mathbb{R}} := \Omega^{2\ell+1}(X)$ of all $2\ell + 1$ -forms.

Euclidean action. To be able to write the Euclidean action in a simple and workable form we need to choose an orthogonal coordinate system $V_{\mathbb{R}} = V_2 \oplus V_2^\perp$ where V_2 is a Lagrangian subspace



and $V_2^\perp = *_E V_2$ is its orthogonal complement with respect to the Hodge metric. From the positivity of the Riemannian metric g_E it follows that $V_2 \cap *_E V_2 = \{0\}$, and thus this orthogonal decomposition is also a Lagrangian decomposition. So any form $v \in V_{\mathbb{R}}$ can be uniquely written in the form $v = v_2 + v_2^\perp$ for some $v_2 \in V_2$ and $v_2^\perp \in V_2^\perp$.

A choice of Lagrangian subspace V_2 defines a Lagrangian subspace $\Gamma_2 \subset H_{DR}^{2\ell+1}(X)$ in the DeRham cohomology. Next, we choose an arbitrary *complementary* Lagrangian subspace $\Gamma_1 \subset H_{DR}^{2\ell+1}(X)$, and define a Lagrangian subspace $V_1 \subset V_{\mathbb{R}}$ by

$$V_1 = \{R \in \Omega_d^{2\ell+1}(X) \mid [R]_{DR} \in \Gamma_1\}. \quad (1.9)$$

Here $\Omega_d^{2\ell+1}(X)$ is the space of all closed $2\ell + 1$ -forms. Note that the subspaces V_1 and V_2 are not complementary: $V_{12} := V_1 \cap V_2 = \{\text{exact forms in } V_2\}$. This fact will result in an extra gauge invariance of the action.

In the orthogonal coordinates $R \in V_1$ can be uniquely written as

$$R = R_2 + R_2^\perp. \quad (1.10)$$

Since R is constrained to be in the Lagrangian subspace V_1 the coordinates R_2 and R_2^\perp are *not* independent. In these coordinates the Euclidian action (1.8) for the imaginary anti-self dual field takes the simple form

$$S_E(R) := \pi \int_X (R_2^\perp \wedge *_E R_2^\perp - i R_2 \wedge R_2^\perp). \quad (1.11)$$

Note that by construction the action vanishes on elements from V_2 . Since V_1 has a nontrivial intersection with V_2 the action (1.11) has an extra gauge symmetry: for any $R \in V_1$ and $v_{12} \in V_1 \cap V_2$ we have $S_E(R + v_{12}) = S_E(R)$.

²Here we ignore possible half-integer shifts in the Dirac quantization law. See below.

It is important that the action (1.11) *depends* on the choice of Lagrangian decomposition, while a properly normalized partition function does not. This comes as follows: the theta function and the normalization factor \mathcal{N}_g in (1.7) transform in metaplectic representations of the group $Sp(b_{2\ell+1}, \mathbb{Z})$ of weight $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Thus modulo the phase factor the partition function (1.7) does not depend on the choice of Lagrangian decomposition. This is the source of much of the difficulty people have had in writing an action principle for the self-dual field.

Lorentzian action. The action on a Lorentzian manifold (M, g) can be obtained from (1.11) by Wick rotation:

$$S_L(R) := \pi \int_M (R_2^\perp \wedge *R_2^\perp + R_2 \wedge R_2^\perp). \quad (1.12)$$

However for this expression to be meaningful we need to require that the Lagrangian subspace V_2 be such that

$$V_2 \cap *V_2 = \{0\}. \quad (1.13)$$

(Otherwise $V_2 \oplus *V_2$ does not define an orthogonal coordinate system on $\Omega^{2\ell+1}(M)$.)

The variation of the action (1.12) with respect to $R \mapsto R + d\delta c$ where $\delta c \in \Omega_{cpt}^{2\ell}(M)$ is

$$\delta S_L(R) = -2\pi \int_M \delta c \wedge d\mathcal{F}^+(R) \quad (1.14)$$

where $\mathcal{F}^+(R) := R_2^\perp + *R_2^\perp$.

As a check on our action, we show in section 8 that the variation of the action with respect to the metric yields the standard stress-energy tensor for the self-dual field \mathcal{F}^+ :

$$\delta_g S_L(R) = \frac{\pi}{2} \int_M (\delta g^{-1} g)^\mu{}_\nu \mathcal{F}^+ \wedge *(dx^\nu \wedge i(\frac{\partial}{\partial x^\mu})\mathcal{F}^+). \quad (1.15)$$

Note that although the action (1.12) depends on $R \in V_1$ its metric variation depends only on $\mathcal{F}^+(R)$. Note also that this result is independent of Lagrangian decomposition. Moreover, as noted in footnote 10 of [27] it offers a promising way to describe the important metric-dependent factor \mathcal{N}_g in the partition function (1.7). We make some preliminary remarks on this topic in section 6.3. One relatively straightforward thing we explain is the norm-square of \mathcal{N}_g . See section 6.3. This follows simply from normalizing the Chern-Simons wavefunction and strengthens the idea, common in AdS/CFT, that the boundary partition function should be identified with the *normalized* bulk wavefunction.

Examples. In section 7.2 we consider the example of a self-dual field on a product manifold $M = \mathbb{R} \times N$ where N is a compact Riemannian $4\ell + 1$ -manifold. On a product manifold there is

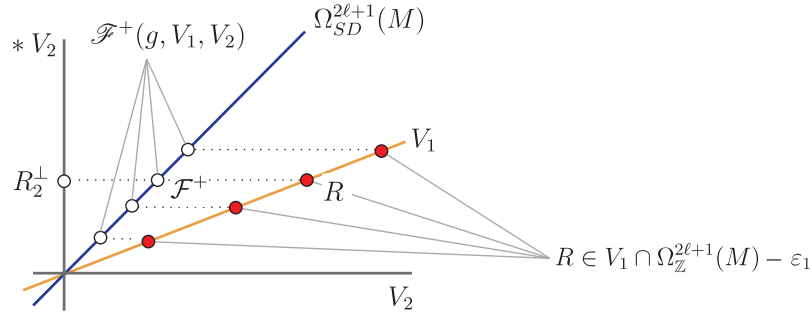
a natural choice of Lagrangian subspaces:

$$V_2 = \Omega^1(\mathbb{R}) \otimes \Omega^{2\ell}(N), \quad \Gamma_2 \cong H_{DR}^{2\ell}(N) \quad \text{and} \quad \Gamma_1 \cong H_{DR}^{2\ell+1}(N).$$

With this choice the action (1.12) for $\ell = 0$ gives an action for the chiral scalar which has appeared previously [5] for $M = \mathbb{R}^2$. For $\ell \geq 1$ it gives the Henneaux-Teitelboim action [9]. Note, however, that our interpretation of the action differs from the one given by Henneaux and Teitelboim. See section 7.3 where we compare the two constructions. We stress that these are just special cases of our general action which can be formulated on arbitrary manifolds with arbitrary metrics.

1.3 Dirac quantization

Let us now return to the second conundrum surrounding (1.3). We see from (1.14) that we should distinguish between R and the self-dual flux $\mathcal{F}^+(R)$. Thus the way out is to understand the quantization condition in a broader sense: there is an abelian group $\mathcal{F}^+(g, V_1, V_2)$ with nontrivial connected components inside the space of *closed* self-dual forms $\Omega_{SD}^{2\ell+1}(M)$, and the classical self-



dual field $\mathcal{F}^+(R)$ takes values only in this group: $\mathcal{F}^+(R) := R_2^\perp + *R_2^\perp$ where $R = R_2^\perp + R_2$ and $[R]_{DR} \in \Gamma_1 - [\varepsilon_1]$.

There is a further subtlety in Dirac quantization involving half-integral shifts related to the characteristics. In order to understand these we must return to one of the subtleties suppressed in our discussion thus far of (1.7). We mentioned above that one must make extra choices even to define a level $k = 1$ Chern-Simons term. These choices enter the theory through the formulation of the Gauss law. In order to write a Gauss law one must choose a $U(1)$ -valued function $\Omega : H^{2\ell+1}(X; \mathbb{Z}) \rightarrow U(1)$ which satisfies the cocycle condition

$$\Omega(a_1 + a_2) = \Omega(a_1)\Omega(a_2)(-1)^{\int_X a_1 \cup a_2}. \quad (1.16)$$

In other words, to specify a Chern-Simons theory one must not only choose a level k , but also

an Ω -function.³ As mentioned above, we will refer to this as a *QRIF*.⁴ For a pure self dual field the Ω functions is \mathbb{Z}_2 -valued. In this case the set of such functions is a principal homogeneous space (torsor) for the 2-torsion points in $H^{2\ell+1}(X; \mathbb{R}/\mathbb{Z})$. The choice of Ω then generalizes the choice of spin structure for the self-dual scalar in two dimensions. The half-integer shifts in the quantization law depends on the choice of Ω . Once a choice of Ω has been made, the level 1 theta function is uniquely fixed. However, in order to write a formula such as (1.7), and thus deduce the shift ε_1 in the quantization law of R , we must also choose a Lagrangian decomposition. Thus, the shift in the quantization law depends on both the QRIF and the Lagrangian decomposition.

The Ω function has certainly appeared in previous treatments of self-dual fields. It was first used in the case of the chiral scalar ($\ell = 0$) in [38, 39], in which case it is related to the mod two index of the Dirac operator. In that treatment the factor was introduced in order to obtain a factorization of a sum over instanton sectors of the nonchiral field into a square of a *single* theta function.⁵ In [27, 28] Witten showed that it plays a central role in the theory of self-dual forms in general. His discussion was based on the general theory of theta functions. He used Ω to define precisely the holonomies of a line bundle with connection over the intermediate Jacobian. An analogous function also plays an important role in the theory of RR fields [40, 41, 34]. One novelty in the present paper is that we show how Ω arises directly in formulating the Gauss law of the Chern-Simons theory.

There is a further subtlety associated with the Ω function. As explained in section 5.2, associated with Ω is a (torsion) characteristic class $\mu \in H^{2\ell+2}(X; \mathbb{Z})$. The Gauss law for the Chern-Simons theory (in the absence of Wilson line sources) says that $F = 0$, where F is the fieldstrength of the Chern-Simons field. However, it also says that the topological component $a(\check{A}) \in H^{2\ell+2}(X; \mathbb{Z})$ of the Chern-Simons field is constrained by the tadpole constraint:

$$ka(\check{A}) + k\mu = 0. \quad (1.17)$$

The class μ has the physical interpretation of being the Poincaré dual of a brane wrapping a torsion cycle. In section 5.2 we explain that when we choose Ω we choose whether branes wrap certain 2-torsion cycles in X . The set of choices of Ω for *fixed* background charge μ is a torsor for $\bar{H}^{2\ell+1}(X; \mathbb{Z})/2\bar{H}^{2\ell+1}(X; \mathbb{Z})$. This defines a set of $2^{b_{2\ell+1}}$ distinct partition functions, generalizing the set of partition functions of the self-dual scalar on a Riemann surface.

³Here is the “end-run” around the Hopkins-Singer theorem mentioned above. Rather than working directly with their Chern-Simons term we instead formulate the theory in terms of Ω , following [27, 28].

⁴QRIF stands for “Quadratic Refinement of the Intersection Form.” Note that a choice of QRIF on a $4\ell + 2$ -manifold X is closely related to a choice of spin structure on the finite dimensional torus $\Omega_d^{2\ell+1}(X)/\Omega_{\mathbb{Z}}^{2\ell+1}(X)$.

⁵The original computation of [38, 39] can be generalized to higher dimensions. The fact that the insertion of Ω changes the splitting of the nonchiral sum over instantons is still overlooked or misunderstood even by quite reputable authors, so we present the details in appendix E.

It is important to note that in the M -theory five-brane the analog of Ω is $U(1)$ valued, not just \mathbb{Z}_2 -valued. Moreover the Ω function is not a choice, but rather is determined by the topology and geometry of the embedded 5-brane and will in general vary continuously with the metric [28, 33].

2 Self-dual field on a $4\ell + 2$ dimensional manifold

In [27] Witten proposed a recipe for constructing a partition function of the chiral 2ℓ -form potential on a $4\ell + 2$ dimensional Riemannian manifold X . In this section we review his basic construction. (We will be slightly more general, introducing a theory depending on two integers p, q .) Consider the Euclidean action for a topologically trivial 2ℓ -form gauge field $C \in \Omega^{2\ell}(X)$ with coupling constant g^2 :

$$S = \frac{1}{2g^2} \int_X dC \wedge *_E dC. \quad (2.1)$$

We introduce now a topologically trivial $U(1)$ external gauge field $A \in \Omega^{2\ell+1}(X)$ with gauge transformation law $\delta A = d\lambda$ and $\delta C = q\lambda$, so that the field C has charge $q \in \mathbb{Z}$ under this $U(1)$. Consider the Lagrangian

$$e^{-S(C,A)} = \exp \left[-\frac{1}{2g^2} \int_X (dC - qA) \wedge *_E (dC - qA) + i\pi p \int_X A \wedge dC \right] \quad (2.2)$$

where p is an integer. For simplicity we will assume that p and q are relatively prime integers.⁶ To understand the effect of the topological interaction it is useful to rewrite this Lagrangian in complex coordinates. Thus, setting $R = dC$ we have $R = R^+ + R^-$ and $A = A^+ + A^-$ where

$$R^\pm = \frac{1}{2}(R \pm i *_E R) \quad (2.3)$$

and similarly for A^\pm . Here A and C are real forms and thus $(A^-)^* = A^+$ and $(R^-)^* = R^+$. In this notation we obtain

$$e^{-S(C,A)} = \exp \left[\frac{i}{g^2} \int_X \left\{ R^- R^+ + q^2 A^- A^+ + (\pi g^2 p + q) A^+ R^- + (\pi g^2 p - q) A^- R^+ \right\} \right].$$

Now one sees that at special values of the coupling constant $g_{p/q}^{-2} = \pi p/q$ only the A^- part of the gauge field A couples to the C field (here we assume that p/q is positive). At this coupling constant the action has the form

$$e^{-S(C,A)} = \exp \left[i\pi \int_X \left\{ \frac{p}{q} R^- R^+ + pq A^- A^+ + 2q A^+ (R^+)^* \right\} \right]. \quad (2.4)$$

⁶If $\gcd(p, q) = m \neq 1$ we can rescale the external gauge field $A \mapsto A' = mA$ and consider the action (2.2) as function of the rescaled gauge field A' . The normalization of A is fixed, so this is really a different theory. In particular, the factorization in appendix E is more subtle.

It follows that the holomorphic dependence of the partition function

$$\mathcal{Z}(A) = \int_{\text{top. trivial.}} \mathcal{D}C e^{-S(C,A)} \quad (2.5)$$

on A^- represents the coupling to the self-dual degree of freedom. Therefore, let us introduce the Hodge complex structure $J = -*_E$ on the space of complexified gauge fields in which A^+ is holomorphic and A^- is antiholomorphic. The covariant derivatives are

$$D^+ = \int_X \delta A^+ \left[\frac{\delta}{\delta A^+} - i\pi k A^- \right] \quad \text{and} \quad D^- = \int_X \delta A^- \left[\frac{\delta}{\delta A^-} - i\pi k A^+ \right] \quad (2.6)$$

where $k = pq$. The partition function obeys the holomorphic equation

$$D^- \mathcal{Z}(A) = 0. \quad (2.7)$$

This easily follows since the Lagrangian (2.4) satisfies this equation. Since $[D^-, D^-] = 0$ the connection (2.6) defines a holomorphic line bundle $\mathcal{L}^{\otimes k}$ over the space of complexified gauge fields A .

The partition function is a holomorphic section of $\mathcal{L}^{\otimes k}$. The fact that the partition function is not a function but a section of a line bundle is related to the fact that the action (2.2) is not gauge invariant. If X is a closed manifold then under the gauge transformation $\delta C = q\lambda$, $\delta A = d\lambda$ it transforms as

$$\delta S = i\pi k \int_X c \wedge F \quad \Rightarrow \quad \mathcal{Z}(A + d\lambda) = \mathcal{Z}(A) e^{i\pi k \int_X \lambda \wedge F} \quad (2.8)$$

where $F = dA$ is the curvature of A . Thus the partition function obeys the non standard gauge-invariance:

$$[dD_A - i\pi k F] \mathcal{Z}(A) = 0$$

where $D = D^+ + D^- = \delta - i\pi k \int_X \delta A \wedge A$ with δ being the differential on the space of gauge fields. The connection D has a nonzero curvature

$$D^2 = -2\pi i k \omega \quad \text{where} \quad \omega = \frac{1}{2} \int_X \delta A \wedge \delta A. \quad (2.9)$$

Complexification of the gauge group. The fact that the partition function is a *holomorphic* section of $\mathcal{L}^{\otimes k}$ allows us to complexify the gauge group. Recall that originally the partition function $\mathcal{Z}(A)$ was a function of a real gauge field A . By writing $A = A^+ + A^-$ we realized that it depends holomorphically on the complex field A^+ . This means that A^+ and A^- can be considered as independent complex variables so $(A^-)^* \neq A^+$. This in turn allows us to complexify the gauge

group. Originally, the gauge transformations were given by a real form $c \in \Omega^{2\ell}(X)$: $A \mapsto A + dc$. Complexification of the gauge group means that now we have two complex gauge parameters c^+ and c^- , and gauge transformations

$$A^+ \mapsto A^+ + \frac{1}{2}(dc^+ + i * dc^+) \quad \text{and} \quad A^- \mapsto A^- + \frac{1}{2}(dc^- - i * dc^-).$$

Notice that the field strength $F = dA^+ + dA^-$ is not invariant under the complex gauge transformation:

$$F \mapsto F + \frac{1}{2i} d * d(c^+ - c^-).$$

Evidently, by a complex gauge transformation we can restrict a topologically trivial gauge field A to be flat, $dA = 0$.

To proceed further we need to modify the partition function (2.5) to include a sum over topological sectors. This step is quite nontrivial, and requires conceptual changes. We postpone the details of the construction to the next section. The partition function takes the schematic form

$$\mathcal{Z}_{p,q}(A) := \sum_{a \in H^{2\ell+1}(X; \mathbb{Z})} (\Omega(a))^k \int_{\text{fixed top. sector}} \mathcal{D}C_a e^{-S(R,A)} \quad (2.10)$$

where $\Omega : H^{2\ell+1}(X; \mathbb{Z}) \rightarrow \{\pm 1\}$ is the crucial phase factor described in the introduction and discussed in detail in section 5.2.

Partition function as a holomorphic section of a line bundle. The space of topologically trivial flat gauge fields is a torus:

$$\mathcal{W}^{2\ell+1}(X) = \Omega_d^{2\ell+1}(X, \mathbb{R}) / \Omega_{\mathbb{Z}}^{2\ell+1}(X) \quad (2.11)$$

which is a quotient of the space of closed fields, $\Omega_d^{2\ell+1}(X)$, by the group of large gauge transformations $A \mapsto A + R$ where R is a closed $2\ell + 1$ -form with integral periods. Thus the partition function is a holomorphic section of the line bundle $\mathcal{L}^{\otimes k}$ over the complex torus $\mathcal{W}_{\mathbb{C}}^{2\ell+1}(X)$ which is obtained from the real torus $\mathcal{W}^{2\ell+1}(X)$ by using the Hodge complex structure J . Thus $\dim_{\mathbb{R}} \mathcal{W}_{\mathbb{C}} = b_{2\ell+1}$. The line bundle $\mathcal{L} \rightarrow \mathcal{W}^{2\ell+1}(X)$ has a nonzero first Chern class $c_1(\mathcal{L}) = k[\omega]_{DR}$. The symplectic form ω is of type $(1, 1)$ in the Hodge complex structure J . From the Kodaira vanishing theorem and the index of $\bar{\partial}$ -operator it follows that

$$\dim H^0(\mathcal{W}_{\mathbb{C}}^{2\ell+1}, \mathcal{L}^{\otimes k}) = \int_{\mathcal{W}^{2\ell+1}} e^{kc_1(\mathcal{L})} \text{Td}(T\mathcal{W}^{2\ell+1}) = \int_{\mathcal{W}^{2\ell+1}} e^{k\omega} = k^g \quad (2.12)$$

where $g = \frac{1}{2} \dim H_{DR}^{2\ell+1}(X)$.

It was argued in [27] that for $k = 1$ this construction describes the partition function of a self-dual field. From (2.12) it follows that the line bundle \mathcal{L} has *unique* holomorphic section. This holomorphic section is the partition function for a self-dual field.

Therefore to construct a partition function for a self-dual particle we need to

1. construct a line bundle \mathcal{L} over the torus $\mathcal{W}^{2\ell+1}(X)$ equipped with norm and hermitian connection ∇ with curvature $-2\pi i\omega$.
2. choose the Hodge complex structure on the torus $\mathcal{W}^{2\ell+1}(X)$. Using the connection $\nabla^{0,1}$ we can define holomorphic sections of \mathcal{L} .

A natural geometrical way of constructing the line bundle and connection on it is to use Chern-Simons theory in one dimension higher.

Relation to Chern-Simons theory. A lot of information about the line bundle $\mathcal{L}^{\otimes k}$ is encoded in the topological term

$$e^{i\pi p \int_X A \wedge dC}. \quad (2.13)$$

Recall that this exponential is not gauge invariant: under the gauge transformation $\delta A = d\lambda$ and $\delta C = q\lambda$ it transforms by

$$e^{i\pi p \int_X A \wedge dC} \mapsto \exp \left[i\pi qp \int_X \lambda \wedge F \right] e^{i\pi p \int_X A \wedge dC} \quad (2.14)$$

This extra phase coming from the gauge transformation looks like the boundary term of a level $k = qp$ “spin” abelian Chern-Simons theory in one dimension higher. Indeed, let Y be a $(4\ell + 3)$ -manifold with boundary X . Consider the following topological action for a topologically trivial gauge field $A \in \Omega^{2\ell+1}(Y)$

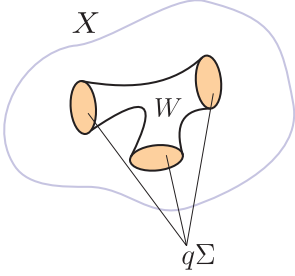
$$e^{2\pi i k \text{CS}_Y(A)} = e^{i\pi k \int_Y A \wedge dA}. \quad (2.15)$$

This Lagrangian is not gauge invariant on a manifold with boundary. Under the gauge transformation $A \mapsto A + d\lambda$ it shifts by the boundary term (2.14). The Chern-Simons functional on a manifold Y with boundary X is most naturally considered as a section of the line bundle \mathcal{L}_{CS} over the space of gauge fields A_X on the boundary X . Our simple calculation shows that \mathcal{L}_{CS} and \mathcal{L} are isomorphic line bundles.

Up to now we were able to identify $k = qp$ with the level of the “spin” abelian Chern-Simons in one dimensional higher. But how do the separate factors p and q appear in the construction? One should think of the self-dual partition functions \mathcal{Z}_{pq}^+ as conformal blocks. There are several ways to construct a correlation function by gluing the conformal blocks: different gluings corresponds to different factorizations of the level k into relatively prime factors $k = pq$.

Now we need to add source terms to the self-dual partition function (see also Appendix A in [35] and section 8.2 of Witten's lectures in [46]). We start again from the action (2.1). Clearly the gauge invariant coupling of the C -field is

$$e^{2\pi i \oint_{\Sigma} C} \quad (2.16)$$



where Σ is a closed 2ℓ -cycle in X . The cycle Σ is not necessarily connected, but might have several connected components. Since X is compact we are forced to assume that $[\Sigma] = 0$ in the homology (the total charge on a compact manifold must be zero). Now we need to generalize this coupling to the gauged theory (2.2). The coupling (2.16) is not gauge invariant: under a gauge transformation $\delta C = q\lambda$ it is multiplied by the factor

$$\exp \left[2\pi i q \oint_{\Sigma} \lambda \right]. \quad (2.17)$$

It is natural to think of the coupling (2.16) as a section of a line bundle over the space of Σ 's. More precisely we choose a cobordism W of q copies of Σ : $\partial W = q\Sigma$ and consider the coupling of the A field

$$\exp \left[2\pi i \int_W A \right]. \quad (2.18)$$

This expression is not gauge invariant under $\delta A = d\lambda$ but multiplies by factor (2.17). Thus the couplings (2.16) and (2.18) are sections of isomorphic line bundles. To interpret the coupling (2.18) in the Chern-Simons theory on Y we just need to push out W from X to Y while keeping the boundary components of W on X so that the embedding $(q\Sigma, W) \hookrightarrow (X, Y)$ is a neat map.

Summary. In this section we argued that the partition function of imaginary self-dual field C , or more general CFT at coupling $g_{p/q}^{-2} = p/q$, can be obtained by quantizing level $k = qp$ spin abelian Chern-Simons theory in one dimension higher. The coupling of the field C to the external sources can be obtained by considering Wilson surfaces in the Chern-Simons theory. For a topologically trivial gauge field A the theory is of the form

$$\exp \left[i\pi k \int_Y A \wedge dA + 2\pi i \int_W A \right] \quad (2.19)$$

where $\partial W = q\Sigma$. To proceed further we need to generalize this action to topologically nontrivial gauge fields A . This is the subject of the next two sections.

3 Field space and gauge transformations

To proceed further we need to generalize the above construction to allow topologically nontrivial gauge fields C and A . The set of gauge-inequivalent fields is an infinite dimensional abelian group $\check{H}^{2\ell+2}(Y)$, known as a Cheeger-Simons cohomology group. For an explanation of this see the pedagogical introduction to Cheeger-Simons cohomology in section 2 of [43]. This group can be described by two exact sequences:

✎ Field strength exact sequence

$$0 \rightarrow \underbrace{H^{2\ell+1}(Y; \mathbb{R}/\mathbb{Z})}_{\text{flat fields}} \rightarrow \check{H}^{2\ell+2}(Y) \xrightarrow{F} \Omega_{\mathbb{Z}}^{2\ell+2}(Y) \rightarrow 0. \quad (3.1)$$

Every differential character \check{A} has a field strength $F(\check{A})$ which is a closed $(2\ell+2)$ -form with integral periods.

✎ Characteristic class exact sequence

$$0 \rightarrow \underbrace{\Omega^{2\ell+1}(Y)/\Omega_{\mathbb{Z}}^{2\ell+1}(Y)}_{\text{topologically trivial}} \rightarrow \check{H}^{2\ell+2}(Y) \xrightarrow{a} H^{2\ell+2}(Y; \mathbb{Z}) \rightarrow 0 \quad (3.2)$$

Every differential character \check{A} has a characteristic class $a(\check{A})$ which is an element of integral cohomology $H^{2\ell+2}(Y; \mathbb{Z})$.

The field strength and characteristic class are compatible in the sense that the reduction \bar{a} of the characteristic class modulo torsion must coincide with the DeRham cohomology class $[F]_{DR}$ defined by the field strength: $\bar{a} = [F]_{DR}$. Putting together the two sequences we can visualize the infinite dimensional abelian group $\check{H}^{2\ell+2}(Y)$ as

$$\check{H}^{2\ell+2}(Y) = \left(\text{torus fibration} \right) \oplus \left(\text{torus fibration} \right) \oplus \dots$$

$\xrightarrow{\quad H^{2\ell+2}(Y; \mathbb{Z}) \quad}$

The group $\check{H}^{2\ell+2}(Y)$ consists of many connected components labeled by the characteristic class $a \in H^{2\ell+2}(Y; \mathbb{Z})$. Each component is a torus fibration over a vector space. The fibres are finite dimensional tori $\mathcal{W}^{2\ell+1}(Y) = \Omega_d^{2\ell+1}(Y)/\Omega_{\mathbb{Z}}^{2\ell+1}(Y)$ represented by topologically trivial flat gauge fields.

There is a product and integration on characters. The product $[\check{A}_1] \cdot [\check{A}_2]$ induces a graded ring structure on $\check{H}^*(Y)$, and the integration $\int^{\check{H}} : \check{H}^{n+1}(X) \rightarrow \mathbb{R}/\mathbb{Z}$ for an n -dimensional manifold Y .

In terms of differential cohomology classes the action of the previous section is generalized to be

$$e^{-S(\check{C}, \check{A})} = \exp \left[-\frac{\pi p}{2q} \int_X (F(\check{C}) - q\check{A}) \wedge *_E(F(\check{C}) - q\check{A}) + i\pi p \int_X \check{A} \cdot \check{C} \right]. \quad (3.3)$$

As in Yang-Mills theory, locality forces one to work with gauge potentials, rather than gauge isomorphism classes of fields. In generalized abelian gauge theories the proper framework is to find a groupoid whose set of isomorphism classes is the set of gauge equivalence classes. The objects in the category are the gauge potentials and the “gauge transformations” are the morphisms between objects. One such groupoid was constructed by Hopkins and Singer, and is known as the groupoid of differential cocycles, denoted by $\check{\mathcal{H}}^{2\ell+2}(Y)$. (We quote the definitions in appendix A). The notation \check{A} is very intuitive: it reminds us that locally \check{A} is described by a differential $(2\ell + 1)$ -form, but the $\check{}$ reminds us that it is not globally well defined.

Unfortunately the category of differential cocycles constructed by Hopkins and Singer involves non-differentiable objects such as real-valued cocycles and is somewhat alien to the intuition of physicists. At the cost of mathematical naturality we will instead postulate that there exists an equivalent category $\check{\mathcal{H}}^{2\ell+2}(X)$ which is closer to the way we think about these objects in physics. We would like our category to be a groupoid obtained by the action of a gauge group on a set of objects.⁷

The gauge group, from which we get the morphisms of the category $\check{\mathcal{H}}^{2\ell+2}(X)$ is, by hypothesis, the group $\check{H}^{2\ell+1}(X)$. The simplest way to motivate this hypothesis is to consider the action of the gauge group on *open* Wilson surfaces on Σ with nonempty boundary. The gauge transformation law should be:

$$\chi_{\check{A}}(\Sigma) \rightarrow \check{\chi}(\partial\Sigma) \chi_{\check{A}}(\Sigma) \quad (3.4)$$

and thus a gauge transformation is precisely given by an element $\check{\chi} \in \check{H}^{2\ell+1}(X)$.

Now, the set of objects of our category forms a space, $\mathcal{C}(X)$. Connected components are labeled by $H^{2\ell+2}(X; \mathbb{Z})$. *We assume that each component can be taken to be a torsor for $\Omega^{2\ell+1}(X)$.* At the cost of naturality, we may choose a basepoint \check{A}_\bullet , and write $\check{A} = \check{A}_\bullet + a$, with $a \in \Omega^{2\ell+1}(X)$. Since $\check{\chi}(\partial\Sigma) = \exp[2\pi i \int_\Sigma F(\check{\chi})]$, it follows that the gauge transformations are given by

$$g_{\check{C}} \check{A}_X = \check{A}_X + F(\check{C}). \quad (3.5)$$

⁷Recall that given any set S and group G acting on S one can form the category $S//G$ whose objects are points in S and whose morphisms are group actions $s \rightarrow s' = g \cdot s$.

Notice that flat characters $H^{2\ell}(X, \mathbb{R}/\mathbb{Z})$ act trivially on the space of gauge fields $\check{\mathcal{H}}^{2\ell+2}(X)$, therefore the group of automorphisms of any object is $\text{Aut}(\check{A}) = H^{2\ell}(X; \mathbb{R}/\mathbb{Z})$.

4 Defining Spin abelian Chern-Simons term in $4\ell + 3$ dimensions

The purpose of this section is to review the Hopkins-Singer definition of a “spin” abelian Chern-Simons term on a $(4\ell + 3)$ -dimensional manifold Y . We use word “spin” loosely here. It does *not* necessarily mean that the manifold Y admits a spin structure, although this is an important special case.

4.1 Chern-Simons functional

First we define a Chern-Simons action on a *closed* manifold Y . For motivation let us begin by assuming that Y is a boundary of a $(4\ell + 4)$ -manifold Z and that the differential character $\check{A} \in \check{\mathcal{H}}^{2\ell+2}(Y)$ extends to a differential character \check{A}_Z defined on Z ⁸. In this case we can define the spin Chern-Simons action by

$$e^{2\pi i k \text{CS}_{\partial Z}(\check{A})} = \exp \left[i\pi k \int_Z F(\check{A}_Z) \wedge F(\check{A}_Z) \right]. \quad (4.1)$$

This expression does not depend on the extension provided that the integral $k \oint_Z F \wedge F$ over any closed $(4\ell + 4)$ -manifold is an *even* integer. This is not always true unless k is an even integer. However from the theory of Wu-classes it follows that

$$a \cup a = \nu_{2\ell+2} \cup a \pmod{2} \quad (4.2)$$

where $a = a(\check{A}_Z)$ is a characteristic class of the differential cocycle \check{A}_Z , and $\nu_{2\ell+2}(Z)$ is the Wu-class of degree $2\ell+2$ on Z ⁹. Thus if the Wu-class $\nu_{2\ell+2}(Z)$ vanishes equation (4.1) indeed defines a topological action. The total Wu class ν is related to the Stiefel-Whitney class w by the Steenrod square operation $w = \text{Sq} \nu$. Using this one easily finds the first few nonzero Wu classes for an orientable manifold

$$\nu_2 = w_2, \quad \nu_4 = w_4 + w_2^2, \quad \nu_6 = w_2 w_4 + w_3^2. \quad (4.3)$$

Actually on an n -dimensional manifold all Wu classes ν_i for $i > [n/2]$ vanish [53]. Thus in particular $\nu_{2\ell+2}(Y) = 0$ for *any* oriented $(4\ell + 3)$ -manifold Y . However it does not necessarily vanish on the extending $(4\ell + 4)$ -manifold Z . Thus the requirement $\nu_{2\ell+2}(Z) = 0$ is a restriction on a choice of Z .

⁸In general there are obstructions to the existence of both Z and the extension \check{A}_Z . In a moment we will define the Chern-Simons functional without appealing to an extension.

⁹For some pedagogical material on the Wu class see [52, 53]

Even in the case when the Wu-class $\nu_{2\ell+2}(Z)$ does not vanish, we can define a topological action. To this end we need to choose *an integral lift* $\bar{\lambda}$ of the Wu-class $\nu_{2\ell+2}$. By an integral lift of the Wu-class we mean an integral cocycle $\bar{\lambda} \in Z^{2\ell+2}(Z; \mathbb{Z})$ such that $\nu_{2\ell+2} = [\bar{\lambda}] \pmod{2}$. Clearly the choice of integral lift is not unique: we can add to $\bar{\lambda}$ any integral cocycle multiplied by 2. Now we can define a Chern-Simons action by

$$e^{2\pi i k CS_{\partial Z}(\check{A})} = \exp \left[2\pi i k \frac{1}{2} \int_Z F(\check{A}_Z) \wedge (F(\check{A}_Z) - \bar{\lambda}_Z) \right]. \quad (4.4)$$

Now we would like to write a formula that does not make use of extensions. Equation (4.4) motivates us to choose a refinement of an integral Wu class, namely a *differential integral Wu class* $\check{\lambda}$. A differential integral Wu class is an element of a category $\check{\mathcal{H}}_V^{2\ell+2}(Y)$ which is a torsor for $\check{\mathcal{H}}^{2\ell+2}(Y)$. The objects are differential cocycles such that $\nu_{2\ell+2} = a(\check{\lambda}) \pmod{2}$. Two choices of differential integral Wu class differ by $\check{\lambda}_1 = \check{\lambda}_2 + 2\check{A}$ for some differential cocycle \check{A} . For further details see appendix A. The fact that (4.4) is well-defined suggests that $\int_Y \check{A} \cdot (\check{A} - \check{\lambda}) \in \mathbb{R}/\mathbb{Z}$ can be divided by two in a well-defined way. It is exactly at this point that a choice of QRIF enters the theory and provides an unambiguous definition of $\frac{1}{2} \int_Y \check{A} \cdot (\check{A} - \check{\lambda}) \in \mathbb{R}/\mathbb{Z}$.

In fact, what Hopkins and Singer take as the basis for their Chern-Simons term is

$$\exp \left[2\pi i k \frac{1}{8} \int_Y (\check{\lambda} \cdot \check{\lambda} - \check{L}_{4\ell+4}) \right] \quad (4.5)$$

where $\check{L}_{4\ell+4}$ is a differential cocycle refining the degree $4\ell + 4$ component of the Hirzebruch L -polynomial¹⁰. Again, having chosen a QRIF the division by 8 is well-defined. This at first seems unrelated to our Chern-Simons term, but if we set $\check{\lambda} = \check{\lambda}_0 - 2\check{A}$ then (4.5) becomes

$$e^{2\pi i k CS_{Y, \check{\lambda}_0}(\check{A})} := \exp \left[2\pi i k \frac{1}{2} \int_Y \check{A} \cdot (\check{A} - \check{\lambda}_0) + 2\pi i k \frac{1}{8} \int_Y (\check{\lambda}_0 \cdot \check{\lambda}_0 - \check{L}_{4\ell+4}) \right] \quad (4.6)$$

In this paper we are mostly concerned with the A -dependence of the Chern-Simons term, but we expect that when one takes into account metric dependence it will be very useful to include the second term. The Chern-Simons functional depends on a choice of $\check{\lambda}$. Its dependence is given by the following simple formulas:

$$CS_{Y, \check{\lambda} - 2\check{B}}(\check{A}) = CS_{Y, \check{\lambda}}(\check{A} + \check{B}) \pmod{1}; \quad (4.7a)$$

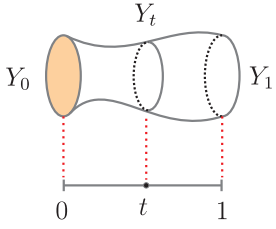
¹⁰Recall that the Hirzebruch polynomial is

$$L = \prod_i \frac{x_i}{\tanh x_i} = 1 - \frac{p_1}{3} + \frac{7p_2 - 2p_1^2}{90} + \dots$$

$$CS(\check{A} + \check{B}) - CS(\check{A}) - CS(\check{B}) + CS(\check{0}) = \int_Y \check{A} \cdot \check{B} \pmod{1}. \quad (4.7b)$$

Thus the Chern-Simons functional is a quadratic refinement of the bilinear form on the objects of $\check{\mathcal{H}}^{2\ell+2}(Y)$.

Variational formula. Suppose we are given a family Z of $4\ell+3$ -manifolds over the interval $[0, 1]$. We denote by Y_t the fibre of this family over point t . Let $\check{A} \in \check{\mathcal{H}}^{2\ell+2}(Z)$ be a differential cocycle and let $\check{\lambda} \in \check{\mathcal{H}}_{\nu_Z}^{2\ell+2}(Z)$ be a differential integral Wu class on the total space of the family Z . Then one can show that the following formula is true



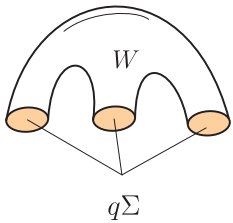
$$CS_{Y_1, \check{\lambda}_1}(\check{A}_1) - CS_{Y_0, \check{\lambda}_0}(\check{A}_0) = \frac{1}{2} \int_Z F(\check{A}) \wedge F(\check{A} - \check{\lambda}) + \frac{1}{8} \int_Z [F(\check{\lambda}) \wedge F(\check{\lambda}) - F(\check{L}_{4\ell+4})] \pmod{1}. \quad (4.8)$$

Chern-Simons functional on a manifold with boundary. On a manifold Y with boundary X the Chern-Simons functional is naturally defined as a section of a line bundle \mathcal{L}_{CS} over the space of gauge inequivalent fields on the boundary $\check{H}^{2\ell+2}(X)$.

The section is constructed as follows: if Y is *any* $(4\ell+3)$ -manifold with boundary X over which the differential cocycles \check{A}_X and $\check{\lambda}_X$ extend then there is a section $\Psi_Y(\check{A})$ of \mathcal{L}_{CS} . Now given two possible extensions (Y_1, \check{A}_1) and (Y_2, \check{A}_2) we must specify the gluing function between them. Let \check{A}_{12} be a differential character on $Y_1 \cup_X \bar{Y}_2$ obtained by combining two extensions \check{A}_1 and \check{A}_2 . Here \bar{Y}_2 denotes manifold Y_2 with opposite orientation. The relation between the sections is

$$\Psi_{Y_1}(\check{A}_1) = e^{2\pi i CS_{Y_1 \cup_X \bar{Y}_2, \check{\lambda}(\check{A}_{12})}} \Psi_{Y_2}(\check{A}_2). \quad (4.9)$$

Differential cocycles on chains. To define the action for topologically nontrivial gauge field we also need to define a holonomy of \check{A} over a chain $W \in Z_{2\ell+1}(Y; \mathbb{Z})$ with boundary $q\Sigma$. The boundary Σ is not necessarily connected but might have several connected components. The holonomy of \check{A} over a chain with boundary is most naturally considered as a section of a line bundle $\mathcal{L}_{q\Sigma}$ over the space of restrictions of gauge field A to Σ .



The section is constructed as follows: if W is *any* $(2\ell+1)$ -chain with the boundary $q\Sigma$ over which the differential cocycles \check{A}_X and $\check{\lambda}_X$ extend then there

is a section $\text{Hol}_W(\check{A})$ of $\mathcal{L}_{q\Sigma}$. Now given two possible extensions W_1 and W_2 we must specify the gluing function between them:

$$\text{Hol}_{W_1}(\check{A}) = e^{2\pi i \check{A}(W_1 \cup_{q\Sigma} \bar{W}_2)} \text{Hol}_{W_2}(\check{A}). \quad (4.10)$$

Chern-Simons functional as a quadratic function. In the previous section we saw that Chern-Simons functional appears in three different kinds: as a number on a closed $4\ell + 4$ manifold; as a map to \mathbb{R}/\mathbb{Z} on a closed $4\ell + 3$ manifold; and as a line bundle with connection on a closed $4\ell + 2$ manifold.

Let E/S be a family of manifolds of relative dimension $4\ell + 4 - i$, with $i \leq 2$. In [29] Hopkins and Singer constructed a Chern-Simons functor from the category of differential integral Wu structures on E to a category $\check{\mathcal{H}}^i(S)$. So depending on i they constructed: for $i = 0$ an integral valued function, for $i = 1$ a function with values in \mathbb{R}/\mathbb{Z} and for $i = 2$ a line bundle with connection. Moreover, these constructions satisfy natural compatibility conditions. We described this construction in a slightly different language in the previous paragraphs.

5 Quantization of the spin abelian Chern-Simons theory

In this section we consider Hamiltonian quantization of the spin abelian Chern-Simons theory on a direct product space $Y = \mathbb{R} \times X$. In general, there are two ways to quantize Chern-Simons theory: one can first impose the equation of motion classically and then quantize the space of solutions of this equation, alternatively one can first quantize the space of all gauge fields and then impose the equation of motion as an operator constraint. In this paper we mostly follow the second approach, although our ultimate goal is to construct wavefunctions on the gauge invariant phase space.

Consider the following topological field theory

$$e^{iS} = e^{2\pi i k CS_{Y,\check{\lambda}}(\check{A})} e^{-2\pi i \check{A}(W)} \quad (5.1)$$

where $W = \mathbb{R} \times q\Sigma$ and $q\Sigma$ is a 2ℓ -dimensional cycle on X . The cycle Σ is not necessarily connected but can have several connected components. Since $\nu_{2\ell+2} = 0$ on Y one can shift \check{A} to choose $\check{\lambda}$ to be flat: $F(\check{\lambda}) = 0$. We will always make this choice.

Using the variational formula (4.8) one obtains the familiar equation of motion

$$kF(\check{A}) = \delta_W \quad (5.2)$$

where δ_W is a $(2\ell + 2)$ -form delta-function supported on the cycle W : $\oint_W a = \int_Y a \wedge \delta_W$.

Hamiltonian. Recall that the set of objects $\mathcal{C}(Y)$ in the category $\mathcal{H}^{2\ell+2}(Y)$ consists of many connected components labeled by the elements of $H^{2\ell+2}(Y; \mathbb{Z})$. Each component is a torsor for $\Omega^{2\ell+1}(Y)$. Thus we can parameterize any gauge field \check{A} with a fixed characteristic class $a(\check{A})$ by choosing a reference gauge field \check{A}_\bullet , then $\check{A} = \check{A}_\bullet + a$ where a is a globally well defined $(2\ell + 1)$ -form. We hope that there will be no confusion between the differential form a and characteristic class $a(\check{A})$. The latter will always come with an argument. Using the variational formula (4.8) one finds the local action:

$$S_{loc}(\check{A}_\bullet + a) = k \int_Y (2\pi a \wedge F_\bullet + \pi a \wedge da) - 2\pi \oint_W a. \quad (5.3)$$

Decomposing

$$a = a_X + dt \wedge a_0 \quad \text{and} \quad F_\bullet = F_{\bullet,X} + dt \wedge F_{\bullet,0} \quad (5.4)$$

where all forms on the right hand side are t -dependent forms on X one finds the local Lagrangian

$$L_{loc}(\check{A}_\bullet + a) = 2\pi k \int_X \left(\frac{1}{2} \dot{a}_X \wedge a_X + F_{\bullet,0} \wedge a_X \right) + 2\pi \int_X a_0 \wedge (kF_X - q\delta_\Sigma). \quad (5.5)$$

Here $F_X = F_{\bullet,X} + da$ and δ_Σ is a $2\ell + 2$ -form delta function supported on Σ : $\oint_\Sigma a_0 =: \int_X a_0 \wedge \delta_\Sigma$.

Phase space. The symplectic form is $2\pi k \omega$ where ω is defined in (2.9). The phase space $P_{q\Sigma}$ is a union of components of $\check{H}^{2\ell+2}(X)$ given by the differential characters with the characteristic class $ka(\check{A}) + k\mu = PD[q\Sigma]$ and (after imposing the classical equation of motion) is isomorphic to a disjoint union of tori modelled on $\mathcal{W}^{2\ell+1}(X) = \Omega_d^{2\ell+1}(X)/\Omega_{\mathbb{Z}}^{2\ell+1}(X)$.

$$P_{q\Sigma} = \{\check{A}_X \in \check{H}^{2\ell+2}(X) \mid ka(\check{A}_X) + k\mu = PD[q\Sigma] \quad \text{and} \quad kF(\check{A}_X) = q\delta_\Sigma\}. \quad (5.6)$$

To quantize Chern-Simons theory we use geometric quantization. Recall that geometric quantization consists of three parts:

1. A choice of prequantum line bundle $\mathcal{L} \rightarrow P_{q\Sigma}$ over the phase space $P_{q\Sigma}$. It must be equipped with norm and hermitian connection with curvature $-2\pi i k \omega$. Then the prequantum Hilbert space $\mathcal{H}^{cl} = L^2(P_{q\Sigma}, \mathcal{L})$ is the space of L^2 -normalizable sections of \mathcal{L} . Note that prequantum line bundle is not unique: one can take $\mathcal{L} \rightarrow \mathcal{L} \otimes S$ where S is a flat unitary line bundle defined by an element in $H^1(P_{q\Sigma}, \mathbb{R}/\mathbb{Z})$.
2. Polarization. If $P_{q\Sigma}$ is a Kähler manifold then there is a natural choice of polarization given by the compatible complex structure J . The quantum Hilbert space $\mathcal{H}_J^{qu} = H_{L^2}^0(\mathcal{L})$ is a subspace of \mathcal{H}^{cl} given by holomorphic sections of \mathcal{L} .
3. The choice of polarization should not be important. The space of quantum Hilbert spaces \mathcal{H}_J^{qu} form a sub-bundle in $\mathcal{H}^{cl} \times \mathcal{T}$ where \mathcal{T} is a Teichmüller space of complex structures on $P_{q\Sigma}$ (see Figure 1). The fact that the quantization is independent of the choice of polarization is made precise by equipping this subbundle with a *projectively flat* connection.

We will approach this by imposing the Gauss law on wavefunctions on an infinite dimensional space.

Prequantum line bundle and connection. The phase space of the Chern-Simons theory is

$$\mathcal{C} := \text{Obj}(\check{\mathcal{H}}^{2\ell+2}(X))$$

the space of gauge fields. The topological action (5.1) defines a natural line bundle

$$\mathcal{L} = \mathcal{L}_{CS}^{\otimes k} \otimes \mathcal{L}_{Hol} \rightarrow \mathcal{C} \times Z_{2\ell}^N. \quad (5.7)$$

We assume that a cocycle $\Sigma \in Z_{2\ell}^N$ consists of N connected components, $\Sigma = n_1 \Sigma_1 + \cdots + n_N \Sigma_N$. Here \mathcal{L}_{CS} is the line bundle corresponding to the level 1 Chern-Simons functional on a manifold with boundary, and \mathcal{L}_{Hol} is a line bundle on which the gauge group $\check{H}^{2\ell+1}(X)$ acts as follows:

$$(g_{\check{C}} \Psi)(\Sigma) = e^{2\pi i q \check{C}(\Sigma)} \Psi(\Sigma). \quad (5.8)$$

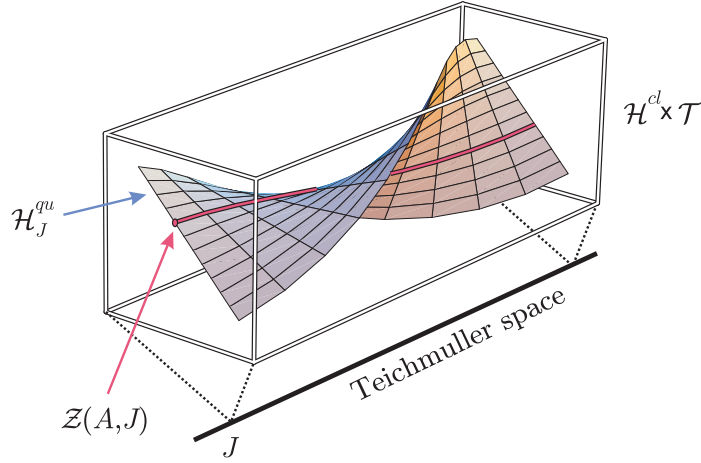


Figure 1: The quantum Hilbert space \mathcal{H}^{qu} is defined as a subbundle inside the trivial bundle $\mathcal{H}^{cl} \times \mathcal{T}$. The fiber \mathcal{H}_J^{qu} over the complex structure J is defined by L^2 -normalizable J -holomorphic sections of $\mathcal{L} \rightarrow P_{q\Sigma}$.

The line bundle $\mathcal{L}_{CS}^{\otimes k}$ has a natural connection defined by the level k Chern-Simons phase. Consider a path $\check{A}_X(t)$ in the space of differential characters \mathcal{C} where $t \in [0, 1]$ is the coordinate on the path. One can think of $\check{A}_X(t)$ as of a differential character from $\check{Z}^{2\ell+2}([0, 1] \times X)$. The parallel transport is defined by

$$\mathcal{U}(\{\check{A}_X(t)\}) := e^{2\pi i k \text{CS}_{[0,1] \times X}(\check{A}_X(t))} \in \text{Hom}(\mathcal{L}_{CS}^{\otimes k}|_{\check{A}_X(0)}, \mathcal{L}_{CS}^{\otimes k}|_{\check{A}_X(1)}). \quad (5.9)$$

The tangent vector to the path $\{\check{A}_X(t)\}$ is $\phi \in \Omega^{2\ell+1}(X)$. The curvature of the connection (5.9) can be computed from the variational formula (4.8):

$$\Omega_{\check{A}_X}(\phi_1, \phi_2) = -2\pi i k \omega(\phi_1, \phi_2) \quad \text{where} \quad \omega(\phi_1, \phi_2) := \int_X \phi_1 \wedge \phi_2. \quad (5.10)$$

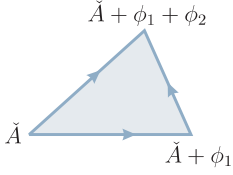
The holonomy function allows one to define a parallel transport of sections of \mathcal{L}_{Hol} along a path $\{\check{A}_X\} \times W_t$ ($t \in [0, 1]$) in the space of 2ℓ -cycles:

$$\mathcal{U}(\{\check{A}_X\} \times W_t) := e^{-2\pi i q \check{A}_X(W)} \in \text{Hom}(\mathcal{L}_{Hol}|_{(\check{A}_X, W_0)}, \mathcal{L}_{Hol}|_{(\check{A}_X, W_1)}). \quad (5.11)$$

The tangent vector to the path W_t is a vector field η defined in the vicinity of W and producing an infinitesimal deformation of the cycle Σ . The curvature of the connection (5.11) at the point $(\check{A}, q\Sigma)$ is

$$\Omega_{(\check{A}_X, \Sigma)}(\eta_1, \eta_2) = 2\pi i q \int_{\Sigma} i_{\eta_1} i_{\eta_2} F(\check{A}_X). \quad (5.12)$$

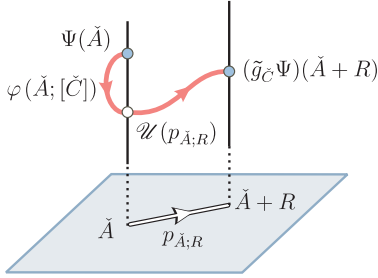
In most equations below we will assume that the 2ℓ -cycle Σ is fixed. So to simplify notations we denote a path $\check{A}_X(t) \times \{\Sigma\}$ in the space of differential characters for the fixed 2ℓ -cycle Σ by $\check{A}_X(t)$.



Now for any $\phi \in \Omega^{2k+1}(X)$ we introduce a straightline path $p_{\check{A};\phi}(t) = \check{A} + t\phi$ in the space of differential characters \mathcal{C} . Using the variational formula (4.8) one finds

$$\mathcal{U}(p_{\check{A}+\phi_1;\phi_2})\mathcal{U}(p_{\check{A};\phi_1}) = e^{-i\pi k\omega(\phi_1,\phi_2)} \mathcal{U}(p_{\check{A};\phi_1+\phi_2}). \quad (5.13)$$

Now we need to lift action of the gauge group (defined in section 3 above) to the line bundle \mathcal{L} . The difference between a group lift and parallel transport is a cocycle. That is, we can define the group lift by



$$(\tilde{g}_{\check{C}}\Psi)(g_{\check{C}}\check{A}_X; \Sigma) := \varphi(\check{A}_X; [\check{C}])e^{-2\pi i q\check{C}(\Sigma)}\mathcal{U}(p_{\check{A}_X;R(\check{C})})\Psi(\check{A}_X; \Sigma) \quad (5.14)$$

provided φ is a phase satisfying the cocycle condition:

$$\begin{aligned} \varphi(g_{\check{C}_1}\check{A}_X; [\check{C}_2])\varphi(\check{A}_X; [\check{C}_1]) \\ = \varphi(\check{A}_X; [\check{C}_1] + [\check{C}_2])e^{-i\pi k\omega(R(\check{C}_1),R(\check{C}_2))}. \end{aligned} \quad (5.15)$$

We will impose an operator constraint on the wave function – the Gauss law – which says $(\tilde{g}_{\check{C}}\Psi)(g_{\check{C}}\check{A}; \Sigma) = \Psi(g_{\check{C}}\check{A}; \Sigma)$ (for more details see section 5.3).

5.1 Construction of the cocycle via a Chern-Simons term

One way to construct a cocycle proceeds using a construction going back to Witten and described in detail in [47]. It makes use of the Chern-Simons term as constructed by Hopkins-Singer and described in the previous section. Since the Hopkins-Singer definition is not explicit enough for our purposes, in the next section we will take a different route to the cocycle. However, the Chern-Simons definition of the cocycle provides an important motivation for the construction we use.

Recall that a Chern-Simons functional on $4\ell + 2$ -manifold X defines a line bundle with connection. The holonomy of this connection around the loops in $\check{H}^{2\ell+2}(X)$ is a natural candidate for the cocycle φ [28]. This holonomy can be calculated as follows [28, 47]: construct a differential cocycle on closed $(4\ell + 3)$ -manifold $S^1 \times X$:

$$\check{A}_X + \check{t} \cdot \check{C}$$

where $[\check{t}] \in \check{H}^1(S^1)$ is the canonical character associated with $S^1 \cong U(1)$. This character has a field strength $F(\check{A}) + dt \wedge R(\check{C})$ and characteristic class $a + [dt] \cup a(\check{C})$. The holonomy on cycles of type $\{t\} \times W_{2\ell+1}$ is

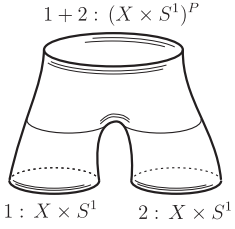
$$e^{2\pi i \check{A}(W_{2\ell+1})} e^{2\pi i t \oint_{W_{2\ell+1}} R(\check{C})},$$

and $e^{2\pi i \check{C}(\Sigma_{2\ell})}$ on cycles of type $S^1 \times \Sigma_{2\ell}$. Now using this twisted differential character we can define

$$\varphi(\check{A}_X; [\check{C}]) := e^{2\pi i k \text{CS}_{S^1 \times X}(\check{A}_X + \check{t} \cdot \check{C})}. \quad (5.16)$$

Two remarks are in order. First, the half-integer level Chern-Simons term was defined by Hopkins and Singer using a different category from what we are using, but it only depends on the isomorphism class of the objects, and hence it can be applied here. Second, the Chern-Simons term depends on a choice of QRIF. That includes in particular a choice of spin structure on the S^1 , which should be the bounding spin structure S^1_- , for reasons we next discuss.

A standard cobordism argument shows that the functional (5.16) satisfies the cocycle relation (5.15). Indeed, consider a differential character on the $(4\ell + 3)$ -dimensional spin manifold $(S^1_- \times X) \cup (S^1_- \times X) \cup (S^1_- \times X)^P$ restricting to $\check{A}_X + \check{t} \cdot \check{C}_1$, $\check{A}_X + R(\check{C}_1) + \check{t} \cdot \check{C}_2$ and $\check{A}_X + \check{t} \cdot (\check{C}_1 + \check{C}_2)$



on the three components (X^P means a change of orientation on a manifold X). Then we choose the extending “generalized spin” $(4\ell + 4)$ -manifold to be $Z = \Delta \times X$ where Δ is a pair of pants bounding the three circles with spin structure restricting to S^1_- on the three components. To be explicit we can choose Δ to be the simplex $\{(t_1, t_2) : 0 \leq t_2 \leq t_1 \leq 1\}$ with identifications $t_i \sim t_i + 1$. We extend the differential character $\check{A}(t_1, t_2) := \check{A}_X + \check{t}_1 \cdot \check{C}_1 + \check{t}_2 \cdot \check{C}_2$ which clearly restricts to the required characters on the boundary. The field

strength of this character is $F_Z = F(\check{A}_X) + dt_1 \wedge R(\check{C}_1) + dt_2 \wedge R(\check{C}_2)$. We therefore can use the variational formula (4.8) to show

$$\varphi(g_{\check{C}_1} \check{A}_X; [\check{C}_2]) \varphi(\check{A}_X; [\check{C}_1]) \varphi^*(\check{A}_X; [\check{C}_1] + [\check{C}_2]) = e^{i\pi k \int_Z F_Z \wedge F_Z} = e^{-i\pi k \omega(R(\check{C}_1), R(\check{C}_2))}.$$

Using properties of the multiplication of differential characters one can rewrite the cocycle (5.16) as

$$\varphi(\check{A}_X; [\check{C}]) = \Omega([\check{C}])^k e^{2\pi i k \int_X [\check{C}] \cdot [\check{A}_X]}. \quad (5.17)$$

From the properties of the Chern-Simons functional we find that $\Omega([\check{C}])$ is a locally constant function of $[\check{C}]$. Therefore it only depends on $a([\check{C}])$. Since there is no difficulty in defining the integral level Chern-Simons term (i.e. k divisible by two), Ω must take values $\{\pm 1\}$. Finally, from the cocycle condition we derive:

$$\Omega([\check{C}_1] + [\check{C}_2]) = \Omega([\check{C}_1]) \Omega([\check{C}_2]) (-1)^{\int_X a(\check{C}_1) \cup a(\check{C}_2)} \quad (5.18)$$

5.2 A direct construction of the cocycle

The definition of the half-integral level Chern-Simons term ($k = 1$ in our notations) in [29] is very subtle, especially in its dependence on certain choices. Therefore, we take a different view here.

Our viewpoint is closer to that of Witten's in [28].

Using the Chern-Simons definition as motivation we construct the cocycle by setting

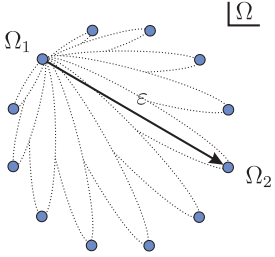
$$\varphi(\check{A}_X; [\check{C}]) = \Omega^k(a(\check{C})) e^{2\pi i k \int_X [\check{C}] \cdot [\check{A}_X]}. \quad (5.19)$$

but now, we seek to *define* the cocycle by *choosing* a function $\Omega : H^{2\ell+1}(X; \mathbb{Z}) \rightarrow U(1)$ such that

$$\Omega(a_1 + a_2) = \Omega(a_1)\Omega(a_2)(-1)^{\int_X a_1 \cup a_2}. \quad (5.20)$$

Any such function Ω can be used to construct a Chern-Simons theory. A choice of Ω is a choice of theory.

What are the possible choices of Ω ? Two solutions of (5.20) differ by a homomorphism from $H^{2\ell+1}(X; \mathbb{Z})$ to \mathbb{R}/\mathbb{Z} . By Poincaré duality it follows that any two solutions Ω_1 and Ω_2 are related by $\Omega_2(a) = \Omega_1(a) e^{i\pi \int a \cup \varepsilon}$ where $\varepsilon \in H^{2\ell+1}(X, \mathbb{R}/\mathbb{Z})$. If we want Ω to take values ± 1 then ε is 2-torsion, i.e. $2\varepsilon = 0$.



Now, associated to Ω is an important invariant. Note that since the bilinear form $\int_X a \cup b$ vanishes on torsion classes, Ω is a homomorphism from $H_{tors}^{2\ell+1}(X; \mathbb{Z})$ to \mathbb{R}/\mathbb{Z} . Since there is a perfect pairing on torsion classes it follows that there is a $\mu \in H_{tors}^{2\ell+2}(X; \mathbb{Z})$ such that

$$\Omega(a_T) = e^{2\pi i T(a_T, \mu)} = e^{2\pi i \int_X \alpha \cup \mu} \quad (5.21)$$

for all torsion classes a_T . In the second equality we have written out the definition of the torsion pairing $T(a, \mu)$, namely, if $a_T = \beta(\alpha)$ where β is the Bockstein map then we can express it as a cup product. If we choose Ω to be \mathbb{Z}_2 -valued then μ is 2-torsion. Note that if $\Omega_2(a) = \Omega_1(a) e^{i\pi \int a \cup \varepsilon}$ then $\mu_2 = \mu_1 + \beta(\varepsilon)$.

Thus, the set of \mathbb{Z}_2 -valued solutions Ω is a torsor for the group of 2-torsion points $G = (H^{2\ell+1}(X, \mathbb{R}/\mathbb{Z}))_2$. The set of solutions with a fixed value of μ is a torsor for the 2-torsion points in the identity component $G_0 = \mathcal{W}_2^{2\ell+1}(X)$. The group G_0 is isomorphic to $\bar{H}^{2\ell+1}(X; \mathbb{Z})/2\bar{H}^{2\ell+1}(X; \mathbb{Z})$ where $\bar{H}^{2\ell+1}(X; \mathbb{Z})$ denotes the reduction of the cohomology group modulo torsion.

It remains to establish the existence of a solution to (5.20). To do this we choose a Lagrangian decomposition $\bar{H}^{2\ell+1}(X; \mathbb{Z}) = \bar{\Gamma}_1 \oplus \bar{\Gamma}_2$ and then define

$$\Omega_{\Gamma_1 \oplus \Gamma_2}(a) := e^{i\pi \int_X \bar{a}^1 \wedge \bar{a}^2} \quad (5.22)$$

where $\bar{a}^1 \in \bar{\Gamma}_1$, $\bar{a}^2 \in \bar{\Gamma}_2$. One easily checks that this is a cocycle. Moreover, it clearly has $\mu = 0$.

For $\mu = 0$ the \mathbb{Z}_2 -valued function Ω can be related to quadratic refinements of the cup product. The Gauss-Milgram sum formula allows one to define a mod 2 invariant of Ω (the Arf invariant)

$$\text{arf}(\Omega) := |G|^{-1/2} \sum_{x \in G} \Omega(x). \quad (5.23)$$

From equation (5.20) it follows that $\text{arf}(\Omega)$ takes values in $\{\pm 1\}$. Thus there are two types of Ω 's depending on the value of the Arf invariant:

even solutions: $\text{arf}(\Omega) = +1$ and **odd** solutions: $\text{arf}(\Omega) = -1$.

Let $\{\Omega_s\}$ be the finite set of all solutions with $\mu = 0$. It is a torsor for G_0 . Denote $q(s) := \text{arf}(\Omega_s)$, now using (5.20) one easily obtains:

$$q(s+x) = q(s) \Omega_s(x), \quad \text{or} \quad \Omega_s(x) = \frac{q(s+x)}{q(s)}. \quad (5.24)$$

This shows that $\Omega(x)$ is a ratio of two mod 2 invariants. Moreover, q is a quadratic refinement of the intersection pairing:

$$\frac{q(s+x+y)q(s)}{q(s+x)q(s+y)} = (-1)^{\int x \cup y} \quad (5.25)$$

As mentioned in the introduction, will refer to a choice of solution Ω as to a choice of QRIF. Note that for $\ell = 0$ the set $\{s\}$ is set of spin structures, and $q(s)$ is the mod 2 index of the Dirac operator corresponding to spin structure s , so a choice of QRIF generalizes the choice of spin structure.

5.3 Quantum Gauss law

The wave function must define a section of the line bundle \mathcal{L} over a component of the space of gauge inequivalent fields $\check{H}^{2\ell+2}(X)$ satisfying the tadpole constraint. In the previous section we constructed a line bundle \mathcal{L} over the objects \mathcal{C} of the category $\check{\mathcal{H}}^{2\ell+2}(X)$. A section Ψ of $\mathcal{L} \rightarrow \mathcal{C}$ descends to a section of \mathcal{L} iff it satisfies the Gauss law constraint

$$\Psi(g_{\check{C}}\check{A}, q\Sigma) = \varphi(\check{A}; [\check{C}]) e^{-2\pi i q \check{C}(\Sigma)} \mathcal{U}(p_{\check{A}; R(\check{C})}) \Psi(\check{A}, q\Sigma). \quad (5.26)$$

Tadpole constraint. We have constructed a line bundle with connection over the space of gauge potentials. Now we would like it to descend to a line bundle with connection over the isomorphism classes of fields. The condition for this is the “tadpole condition,” which in our context is the condition that the automorphism group of an object acts trivially on the fiber of the line bundle. (Here we are following the general line of reasoning of [47].) This amounts to the condition

$$\varphi(\check{A}; [\check{C}]) e^{-2\pi i q \check{C}(\Sigma)} = 1$$

on flat characters $[\check{C}] \in H^{2\ell}(X; \mathbb{R}/\mathbb{Z})$. Combining (5.19) with (5.21) we obtain the condition:

$$k\mu + ka(\check{A}) = PD[q\Sigma] \quad (5.27)$$

where $PD[\Sigma]$ is the Poincaré dual to the homology class $[\Sigma] \in H_{2\ell}(X; \mathbb{Z})$.

What is the physical interpretation of μ ? If we view $PD[q\Sigma] - k\mu$ together then we see that we can interpret μ as the class of a background brane wrapping a two-torsion cycle. For example, in the case of the 5-brane partition function, nonzero μ means that the background contains a string wrapping a cycle Poincaré dual to μ .

Thus, from the physical point of view we should first choose μ . This partially specifies the background — telling us the homology class of possible torsion branes. Having fixed that background, the set of possible partition functions is a torsor for G_0 . These are the possible partition functions generalizing the well-known set of partition functions of a self-dual scalar on a Riemann surface.

Finally, let us remark that if $k > 1$ there might be several solutions to (5.27), e.g., if there is k -torsion. In this case the partition function becomes a section of a line bundle $\mathcal{L}^{\otimes k}$ over several connected components.

The cocycle φ looks particularly simple for differential characters satisfying the tadpole constraint (5.27). We choose a flat differential cocycle $[\check{\mu}] \in \check{H}^{2\ell+2}(X)$ with (torsion) characteristic class $a(\check{\mu}) = \mu$. If $\Omega([\check{C}])$ is \mathbb{Z}_2 -valued then one can choose $\check{\mu}$ to be 2-torsion. Now we can rewrite the phase in (5.26) in the form

$$\varphi(\check{A}; [\check{C}]) e^{-2\pi i q \check{C}(\Sigma)} = [\Omega(a(\check{C})) e^{-2\pi i \int_X [\check{C}] \cdot [\check{\mu}]}]^k e^{2\pi i k \int_X [\check{C}] \cdot ([\check{A}] + [\check{\mu}] - \frac{q}{k} [\check{\delta}(\Sigma)])}.$$

Notice that the first term defines a QRIF with zero characteristic class which we denote by $\Omega_0([R(\check{C})])$. It depends only on the DeRham cohomology class of the curvature $R(\check{C})$ of the differential cocycle $[\check{C}]$. For \check{A} satisfying the tadpole constraint (5.27) the term $[\check{A}] + [\check{\mu}] - \frac{q}{k} [\check{\delta}(\Sigma)]$ in the exponential is a topologically trivial differential character. Thus it can be represented by a $2\ell + 1$ form $\sigma(\check{A}, \Sigma)$ which satisfies two properties

$$d\sigma(\check{A}, \Sigma) = F(\check{A}) - \frac{q}{k} \delta(\Sigma) \quad \text{and} \quad \sigma(\check{A} + a, \Sigma) = \sigma(\check{A}, \Sigma) + a \quad \forall a \in \Omega^{2\ell+1}(X). \quad (5.28)$$

So finally the Gauss law (5.26) can be written as

$$\Psi(g_{\check{C}} \check{A}, q\Sigma) = \Omega_0([R])^k e^{-2\pi i k \omega(R, \sigma(\check{A}, \Sigma))} \mathcal{U}(p_{\check{A}, R(\check{C})}) \Psi(\check{A}, q\Sigma). \quad (5.29)$$

To conclude: When the tadpole condition (5.27) is satisfied the line bundle \mathcal{L} with connection descends to a line bundle \mathcal{L} with connection over a component (or components) of $\check{H}^{2\ell+2}(X)$. This line bundle with connection is completely determined by a pair $(\varphi(\check{A}, [\check{C}]), k\omega)$ where ω is the curvature of the connection (a closed 2-form with integral periods) and φ is a cocycle

$$\varphi(\check{A}; [\check{C}_1] + [\check{C}_2]) = \varphi(\check{A} + R_1; [\check{C}_2]) \varphi(\check{A}; [\check{C}_1]) e^{i\pi k \omega(R_1, R_2)} \quad (5.30)$$

which equals 1 on the flat characters $H^{2\ell}(X; \mathbb{R}/\mathbb{Z})$. The cocycle fixes the holonomy of the connection along noncontractible curves in $\check{H}^{2\ell+2}(X)$. The noncontractible curves are specified by elements $[\check{C}]$. Note that the cocycle φ depends both on \check{A} and $[\check{C}]$ as it must since the curvature is nonzero. Notice that if we are in the topologically trivial component, e.g. $a(\check{A}) = 0$, then we have a preferred point $\check{A} = 0$. Then essentially all nontrivial information is encoded in the holonomies through the curves containing the origin [28].

Gauss law in local coordinates. Each component in the space of objects in $\check{\mathcal{H}}^{2\ell+2}(X)$ is a contractible space. Thus the line bundle $\mathcal{L} \rightarrow \mathcal{C}$ is trivial. To construct a section explicitly we need to choose an explicit trivialization of this line bundle. To this end we choose an arbitrary reference differential character \check{A}_\bullet satisfying the tadpole constraint (5.27). Then an arbitrary field configuration (in the connected component) can be parameterized by $\check{A} = \check{A}_\bullet + a$ where a is a globally well defined $(2\ell + 1)$ -form. Define a canonical nowhere vanishing section S of unit norm by

$$S(\check{A}) := \mathcal{U}(p_{\check{A}_\bullet; \check{A} - \check{A}_\bullet}) S_\bullet$$

where $S_\bullet \in \mathbb{C}$ and $|S_\bullet| = 1$. The wave function $\mathcal{Z}_{p,q}(a, \Sigma)$ is a ratio of two sections $\Psi(\check{A})/S(\check{A})$.

From equations (5.14) and (5.29) it follows that the Gauss law takes the following form on the wave function

$$\mathcal{Z}_{p,q}(a + R, \Sigma) = \Omega_0([R])^k e^{-2\pi i k \omega(R, \sigma(\check{A}_\bullet, \Sigma)) - i\pi k \omega(R, a)} \mathcal{Z}_{p,q}(a, \Sigma) \quad (5.31)$$

for an arbitrary closed $(2\ell + 1)$ -form R with integral periods. The form $\sigma(\check{A}, \Sigma)$ is defined in (5.28).

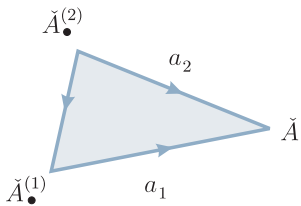
Dependence on a choice of a base point. Recall that the wave function $\mathcal{Z}_{p,q}(a)$ is defined as a ratio of two sections $\Psi(\check{A})$ and $S(\check{A})$. The section S depends on a choice of the base point \check{A}_\bullet , and thus the wave function $\mathcal{Z}_{p,q}$ also depends on this choice.

Let $\check{A}_\bullet^{(1)}$ and $\check{A}_\bullet^{(2)}$ be two base points which solve the tadpole constraint (5.27). Depending on the choice of a base point we have two different trivializing sections S_1 and S_2 :

$$S_i(\check{A}) := \mathcal{U}(p_{\check{A}_\bullet^{(i)}; a_i}) S_\bullet^{(i)}$$

for $i = 1, 2$ and $a_i := \check{A} - \check{A}_\bullet^{(i)}$. The coordinates a_1 and a_2 are related by $a_2 = g_{12}(a_1) := a_1 + \Delta A_\bullet$ where $\Delta A_\bullet := \check{A}_\bullet^{(1)} - \check{A}_\bullet^{(2)}$. In turn the wave functions $\mathcal{Z}_1(a_1)$ and $\mathcal{Z}_2(a_2)$ are related by

$$\mathcal{Z}_2(g_{12}(a_1)) = \tilde{g}_{12}(a_1) \mathcal{Z}_1(a_1) \quad (5.32)$$



where the gluing function \tilde{g}_{12} is

$$\tilde{g}_{12}(a_1) := \frac{S_1(\check{A})}{S_2(\check{A})} = e^{i\pi k \omega(a_1, \Delta A_\bullet)} (S_\bullet^{(2)})^{-1} \mathcal{U}(p_{\check{A}_\bullet^{(1)}, \Delta A_\bullet}) S_\bullet^{(1)}. \quad (5.33)$$

The dependence on a choice of base point arises because there is no canonical trivialization, and is related to the fact that the line bundle is nontrivial over the gauge invariant phase space. Indeed, one interpretation of these formulas is as follows: one can think of a choice of basepoint \check{A}_\bullet and trivializing section $S(\check{A})$ as a choice of local coordinate system on the line bundle $\mathcal{L} \rightarrow \mathcal{C}$ (or \mathcal{L} over a component of $\check{H}^{2\ell+2}(X)$). Once we have chosen \check{A}_\bullet we can identify the space of gauge fields with $\Omega^{2\ell+1}(X)$ (or with $\Omega^{2\ell+1}(X)/\Omega_{\mathbb{Z}}^{2\ell+1}(X)$ respectively). In local coordinates a section $\Psi(\check{A})$ is described by a function $\mathcal{Z}_{p,q}(a)$ where $a := \check{A} - \check{A}_\bullet \in \Omega^{2\ell+1}(X)$. Suppose we are given two coordinate systems $(\check{A}_\bullet^{(1)}, S_1)$ and $(\check{A}_\bullet^{(2)}, S_2)$. The formula (5.32) defines the gluing function \tilde{g}_{12} of local sections \mathcal{Z}_1 and \mathcal{Z}_2 in coordinate systems 1 and 2. One can easily verify that $\tilde{g}_{12}\tilde{g}_{23}\tilde{g}_{31} = 1$, and thus (5.33) defines a cocycle. The cocycle $g_{12}(a_1)$ is globally well defined on $\mathcal{L} \rightarrow \mathcal{C}$, but it does not descend to a globally well defined cocycle on $\mathcal{L} \rightarrow \check{H}^{2\ell+2}(X)$.

6 Construction of the partition function

The content of this section is as follows: To obtain a quantum Hilbert space we need to choose a polarization on the phase space $\mathcal{C} = \text{Obj}(\mathcal{H}^{2\ell+2}(X))$. A choice of Riemannian metric g_E on X defines a complex structure $J = -*_E$ on $T_A\mathcal{C} \cong \Omega^{2\ell+1}(X)$. The quantum Hilbert space consists of holomorphic sections $\{\Psi\}$ of \mathcal{L} , i.e., which satisfy $D^-\Psi = 0$.

Note that there are infinitely many sections of \mathcal{L} which satisfy the Gauss law (5.26), in contrast there are finitely many *holomorphic* sections which satisfy the Gauss law (5.26). By choosing a local coordinate system $(\check{A}_\bullet, S(\check{A}))$ on $\mathcal{L} \rightarrow \mathcal{C}$ one can try to construct a holomorphic solution of the Gauss law explicitly. The resulting expression will in addition depend on some extra choices such as a lagrangian decomposition $\bar{H}^{2\ell+1}(X; \mathbb{Z}) = \bar{\Gamma}_1 \oplus \bar{\Gamma}_2$ of the integral cohomology modulo torsion. The (local) expression for the partition function is summarized by Theorem 6.1.

6.1 Choice of polarization

Equation (5.27) constrains the connected component in the space of the gauge fields \mathcal{C} . Now by choosing a local coordinate system (\check{A}_\bullet, S) we can identify the phase space with the real vector space $V_{\mathbb{R}} = \Omega^{2\ell+1}(X, \mathbb{R})$ by $\check{A} = \check{A}_\bullet + a$, $a \in V_{\mathbb{R}}$.¹¹

The vector space $V_{\mathbb{R}}$ has a natural antisymmetric form

$$\omega(\phi_1, \phi_2) = \int_X \phi_1 \wedge \phi_2. \quad (6.1)$$

This 2-form is closed and nondegenerate and thus it defines a symplectic structure on the space of gauge fields \mathcal{C} . Moreover a choice of Riemannian metric g_E on X defines the Hodge metric on $V_{\mathbb{R}}$


$$g(\phi_1, \phi_2) = \int_X \phi_1 \wedge *_E \phi_2. \quad (6.2)$$

Each metric on $V_{\mathbb{R}}$ defines for us a compatible complex structure:


$$g(\phi_1, \phi_2) = \omega(J \cdot \phi_1, \phi_2) \quad \Rightarrow \quad J \cdot \phi = -*_E \phi. \quad (6.3)$$

Using this complex structure we decompose the space of real forms $V_{\mathbb{R}}$ as

$$V_{\mathbb{R}} \otimes \mathbb{C} \cong V^+ \oplus V^-.$$



imaginary anti self-dual



imaginary self-dual

¹¹If we took the view of constraining and then quantizing then the phase space will be a quotient of the space of closed forms.

Any vector R^+ of the complex vector space V^+ can be *uniquely* written as

$$R^+ = \frac{1}{2}(R + i *_E R) \quad (6.4)$$

for some real vector $R \in V_{\mathbb{R}}$.

This decomposition introduces complex coordinates on the patch (\check{A}_{\bullet}, S) . Recall that in *real* local coordinates we have a covariant derivative $D := \delta - i\pi k \omega(\delta a, a)$ which is defined on sections of the line bundle \mathcal{L} . Here δ is the usual differential with respect to a . One can verify that this definition of the covariant derivative is consistent with the coordinate transformation (5.32). In *complex* coordinates the covariant derivative D decomposes as $D = D^+ + D^-$ where

$$D^+ = \delta^+ - i\pi k \omega(\delta a^+, a^-) \quad \text{and} \quad D^- = \delta^- - i\pi k \omega(\delta a^-, a^+). \quad (6.5)$$

The quantum Hilbert space consists of holomorphic section, i.e. $D^-\Psi = 0$.

In the local coordinates (\check{A}_{\bullet}, S) one can identify holomorphic sections $D^-\mathcal{Z}_{p,q}(a^+, a^-) = 0$ with holomorphic functions $\vartheta(a^+)$ via

$$\mathcal{Z}_{p,q}(a^+, a^-; \Sigma) = e^{i\pi k \omega(a^-, a^+)} \vartheta(a^+; \Sigma). \quad (6.6)$$

Again one can verify that the corresponding gluing functions (5.33) for ϑ depend *holomorphically* on a^+ . In this case the Gauss law constraint (5.31) takes the following simple form

$$\vartheta(a^+ + R^+; \Sigma) = \left\{ \Omega_0([R]) e^{-2\pi i \omega(R, \sigma(\check{A}_{\bullet}, \Sigma))} \right\}^k e^{\frac{\pi k}{2} H(R^+, R^+) + \pi k H(a^+, R^+)} \vartheta(a^+; \Sigma) \quad (6.7)$$

for all $R \in \Omega_{\mathbb{Z}}^{2\ell+1}(X)$. Here we have introduced a hermitian form H on $V^+ \times V^+$. It is defined using the metric g and symplectic form ω :

$$H(u^+, v^+) := 2i \omega(u^+, \overline{v^+}) = g(u, v) + i \omega(u, v). \quad (6.8)$$

In our notation H is \mathbb{C} -linear in the first argument and \mathbb{C} -antilinear in the second: $H(u, v) = \overline{H(v, u)}$.

6.2 Partition function

Equation (6.7) looks like a functional equation for a theta function. The important difference is that the equation for a theta function is usually defined on a finite dimensional vector space, while our equation is on the infinite dimensional vector space $\Omega^{2\ell+1}(X)$. Nevertheless we can use the same technique to solve it, namely, we will use Fourier analysis.

Note that the function $\vartheta(a^+; \Sigma)$ is not invariant under translation $a^+ \mapsto a^+ + R^+$. To be able to apply Fourier analysis we need to have a function which is essentially invariant (i.e. transforms

by a character) under translation of at least “half” of the group $\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. In the theory of theta functions this problem is usually solved by introducing a \mathbb{C} -bilinear form B on $V^+ \times V^+$. Using this \mathbb{C} -bilinear form we can define a new holomorphic function

$$\tilde{\vartheta}(a^+; \Sigma) = e^{-\frac{\pi k}{2} B(a^+, a^+)} \vartheta(a^+; \Sigma) \quad (6.9)$$

which satisfies the following Gauss law

$$\tilde{\vartheta}(a^+ + R^+; \Sigma) = \left\{ \Omega_0([R]) e^{-2\pi i \omega(R, \sigma(\tilde{A}_\bullet, \Sigma))} \right\}^k e^{\frac{\pi k}{2} (H-B)(R^+, R^+) + \pi k (H-B)(a^+, R^+)} \tilde{\vartheta}(a^+; \Sigma). \quad (6.10)$$

The form $(H-B)(\cdot, R^+)$ vanishes on the “half” of the group $\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. Thus we can solve equation (6.10) by the Fourier analysis. So to proceed further we need to define a \mathbb{C} -bilinear form on an infinite dimensional space $V^+ \times V^+$. For simplicity we henceforth put $k = 1$.

Defining a \mathbb{C} -bilinear form. Recall that we are given an infinite dimensional vector space $V_{\mathbb{R}} = \Omega_{\mathbb{Z}}^{2\ell+1}(X)$, a complex structure on it $J = -*_E$ and a symplectic form ω . In the previous subsection using these structures we defined a complex vector space V^+ together with the hermitian form H on it which is \mathbb{C} -linear in the first argument and \mathbb{C} -antilinear in the second.

To define a \mathbb{C} -bilinear form on $V^+ \times V^+$ it is sufficient to have a \mathbb{C} -antilinear involution of V^+

$$\tilde{I} : V^+ \rightarrow V^+ \quad \text{and} \quad \tilde{I}(zv^+) = \bar{z} \tilde{I}(v^+) \quad \forall z \in \mathbb{C}.$$

Indeed, given \tilde{I} we can define the bilinear form by

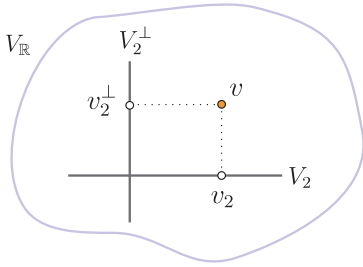
$$B(u^+, v^+) := H(u^+, \tilde{I}(v^+)). \quad (6.11)$$

There is *no* natural choice of such an involution. However a choice of Lagrangian subspace $V_2 \subset V_{\mathbb{R}}$ defines one. Note that $V_{\mathbb{R}} = V_2 \oplus V_2^\perp$ with $V_2^\perp := J(V_2)$ is a Lagrangian decomposition. Indeed, both V_2 and $J(V_2)$ are maximally isotropic subspaces. Thus to prove the statement it is sufficient to show that $V_2 \cap J(V_2) = \{0\}$. Since V_2 is Lagrangian $V_2 \cap J(V_2)$ is a subspace of $V_{\mathbb{R}}$ on which ω vanishes. But J is a compatible complex structure, therefore the metric g defined by (6.3) also vanishes. In turn g , being the Hodge metric, is nondegenerate and thus

$$V_2 \cap J(V_2) = \{0\}.$$

Every Lagrangian decomposition defines an involution I of $V_{\mathbb{R}}$. Any vector $v \in V_{\mathbb{R}}$ can be uniquely written as $v = v_2 + v_2^\perp$. The involution I is defined by changing sign of v_2^\perp

$$I(v) := v_2 - v_2^\perp. \quad (6.12)$$



This involution is compatible with the symplectic structure in a sense that the \mathbb{R} -bilinear form $\omega(I\cdot, \cdot)$ is symmetric. Moreover, it anticommutes with the complex structure, $I \circ J + J \circ I = 0$. The involution I on $V_{\mathbb{R}}$ defines a \mathbb{C} -antilinear involution \tilde{I} on V^+ by¹²

$$\tilde{I}(v^+) := (I(v))^+. \quad (6.13)$$

The form $(H - B)(u^+, v^+)$ can be written in the following two equivalent ways:

$$(H - B)(u^+, v^+) = 2i\omega(u, \mathcal{F}^-(v)) \quad \text{where} \quad \mathcal{F}^-(v) := v_2^\perp - i *_E v_2^\perp; \quad (6.14a)$$

$$= 2g(u_2^\perp, v_2^\perp) + 2i\omega(u_2, v_2^\perp). \quad (6.14b)$$

Or put differently the form B defined by (6.11) has the following properties

$$B|_{V^+ \times V_2^+} = H|_{V^+ \times V_2^+} \quad \text{and} \quad B|_{V_2^+ \times V^+} = H|_{V_2^+ \times V^+} - 2i\omega|_{V_2 \times V_{\mathbb{R}}}. \quad (6.15)$$

From this definition of B it follows that it is completely determined by the hermitian form H and a choice Lagrangian subspace V_2 .

Another way to define B proceeds as follows. Note that V_2^+ is a real subspace of V^+ generating V^+ as a complex vector space. The hermitian form H restricted to $V_2^+ \times V_2^+$ defines a symmetric \mathbb{R} -bilinear form. Its \mathbb{C} -bilinear extension to $V^+ \times V^+$ yields the form B defined above.

Decomposition of $\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. Having chosen B we can now try to solve equation (6.10). From (6.15) it follows that $(H - B)(\cdot, R^+)$ vanishes for $R \in V_2 \cap \Omega_{\mathbb{Z}}^{2\ell+1}(X)$. Thus the function $\tilde{\vartheta}(a^+)$ is essentially invariant (i.e. transforms by a character) under translations by the group $V_2 \cap \Omega_{\mathbb{Z}}^{2\ell+1}(X)$. Now we need to choose a complementary part of this subgroup inside $\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. The complication here is that $\Omega_{\mathbb{Z}}^{2\ell+1}(X)$ is *not* a lagrangian subspace.

We define a “complementary part” of this subgroup as follows. The symplectic form ω on $V_{\mathbb{R}}$ defines a symplectic form on the DeRham cohomology $\Gamma = H_{DR}^{2\ell+1}(X)$. This symplectic form is integral valued on the image $\bar{\Gamma} := \bar{H}^{2\ell+1}(X; \mathbb{Z})$ of integral cohomology inside the DeRham cohomology. In turn, a choice of Lagrangian subspace $V_2 \subset V_{\mathbb{R}}$ defines a Lagrangian subspace $\Gamma_2 \subset H_{DR}^{2\ell+1}(X)$. We denote by choose $\bar{\Gamma}_2$ the corresponding lattice inside $\bar{H}^{2\ell+1}(X; \mathbb{Z})$. Now define Γ_1 to be an arbitrary complementary Lagrangian subspace of Γ_2 such that the lattice $\bar{\Gamma}$ decomposes as $\bar{\Gamma}_1 \oplus \bar{\Gamma}_2$. Choose a subspace $V_1 \subset V_{\mathbb{R}}$ to consist of all closed $2\ell + 1$ -forms whose DeRham cohomology class lies in Γ_1 :

$$V_1 = \{R \in \Omega_d^{2\ell+1}(X) \mid [R]_{DR} \in \Gamma_1\}. \quad (6.16)$$

The intersection $V_1 \cap \Omega_{\mathbb{Z}}^{2\ell+1}(X)$ we denote by \bar{V}_1 , i.e. the space (6.16) with Γ_1 changed to $\bar{\Gamma}_1$.

¹²To prove this note that for $z = x + iy$ we can write $zv^+ = (x + yJ)v^+ = ((x + yJ)v)^+$. Now using definition (6.12) of the involution I one easily verify $\tilde{I}(zv^+) = \bar{z}\tilde{I}(v^+)$.

Lemma 6.1. V_1 defined by (6.16) is a Lagrangian subspace of $\Omega^{2\ell+1}(X)$.

Proof. Note that by construction V_1 is isotropic. We need to prove that V_1 is maximally isotropic. To this end we choose an arbitrary Riemannian metric h on X . Then we can rewrite the definition (7.5) by changing Γ_1 to Γ_1^h . In this form the statement of the lemma obviously follows from the Hodge decomposition. \square

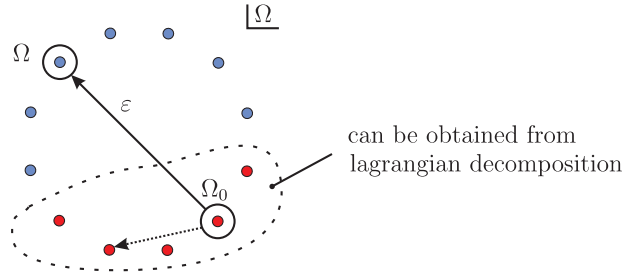
Note that V_1 and V_2 are *not complementary* lagrangian subspaces. They have nonzero intersection $V_{12} := V_1 \cap V_2$ where

$$V_{12} = \{\text{exact forms in } V_2\}. \quad (6.17)$$

Quadratic function Ω_0 . To write down an explicit solution to the Gauss law (6.7) we need a simple formula for Ω_0 . As we have mentioned above a choice of Ω_0 with $\mu = 0$ is naturally determined by a Lagrangian decomposition of $\bar{H}^{2\ell+1}(X; \mathbb{Z}) = \bar{\Gamma}_1 \oplus \bar{\Gamma}_2$. Any $R \in \Omega_{\mathbb{Z}}^{2\ell+1}(X)$ can be written as $R = R_1 + R_2$ where $R_1 \in \bar{V}_1$ and $R_2 \in V_2 \cap \Omega_{\mathbb{Z}}^{2\ell+1}(X)$. Since $V_1 \cap V_2 \neq \{0\}$ this decomposition is not unique and two different decompositions are related by adding exact forms in V_2 . Now define

$$\Omega_{\Gamma_1 \oplus \Gamma_2}(R) := e^{i\pi\omega(R_1, R_2)}. \quad (6.18)$$

Since R_1 and R_2 are closed it follows that $\Omega_{\Gamma_1 \oplus \Gamma_2}(R)$ does not depend on a particular choice of decomposition $R = R_1 + R_2$. Moreover $\Omega_{\Gamma_1 \oplus \Gamma_2}$ takes values in $\{\pm 1\}$.



Given $\Omega_{\Gamma_1 \oplus \Gamma_2}$ we can parameterize all solutions with $\mu = 0$ by $[\varepsilon] \in \Omega_d^{2\ell+1}(X)/\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. So Ω_0 can be written as

$$\Omega_0(R) = e^{i\pi\omega(R_1, R_2) + 2\pi i\omega(\varepsilon, R)}. \quad (6.19)$$

If we want $\Omega_0(R)$ to take values in $\{\pm 1\}$ then ε is quantized $[\varepsilon] \in \Omega_{\frac{1}{2}\mathbb{Z}}^{2\ell+1}(X)/\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. In this case a simple calculation shows that only *even* solutions ($\text{arf}(\Omega) = +1$) can be obtained by a choice of Lagrangian decomposition.

We will see that a choice of ε yields a half integral shift in the flux quantization condition. In particular, if Ω is an odd solution then the self-dual flux is half-integrally quantized.

Partition function. Now one can solve equation (6.10) via Fourier analysis. The expression for the partition function can be summarized by the following theorems:

Theorem 6.1. *The following Euclidean functional integral*

$$\begin{aligned} \vartheta^\eta(a^+) = \exp\left[-\frac{\pi}{2}(H-B)(\eta^+, \eta^+) + \frac{\pi}{2}B(a^+, a^+) - \pi(H-B)(a^+, \eta^+)\right] \\ \times \int_{\bar{V}_1/V_{12}} \mathcal{D}R \exp\left[-\frac{\pi}{2}(H-B)(R^+, R^+) + \pi(H-B)(a^+ + \eta^+, R^+)\right] \end{aligned} \quad (6.20)$$

(where the integral goes over all closed forms $R \in \bar{V}_1$ modulo exact forms in V_2) $a, \eta \in \Omega^{2\ell+1}(X)$ satisfies the functional equation

$$\vartheta^\eta(a^+ + \lambda^+) = \Omega_{\Gamma_1 \oplus \Gamma_2}(\lambda) e^{2\pi i \omega(\eta, \lambda)} e^{\pi H(a^+, \lambda^+) + \frac{\pi}{2} H(\lambda^+, \lambda^+)} \vartheta^\eta(a^+) \quad (6.21)$$

for all $\lambda \in \Omega_{\mathbb{Z}}^{2\ell+1}(X)$.

Remark 6.1. The form $(H-B)$ restricted to $V_1^+ \times V_1^+$ is symmetric. Indeed, the first term in equation (6.14b) is obviously symmetric. For u and v from a Lagrangian subspace we have $0 = \omega(u, v) = \omega(u_2, v_2^\perp) + \omega(u_2^\perp, v_2)$, which shows that the second term in (6.14b) is also symmetric. From (6.14b) it also follows that it vanishes on V_{12} and the $\text{Re}(H-B)|_{V_1^+ \times V_1^+}$ is positive definite on the complement of V_{12} inside V_1 . In the theory of theta functions the quadratic form $(H-B)$ restricted to the finite dimensional space $\Gamma_1^h := V_1 \cap \mathcal{H}^{2\ell+1}(X)$

$$\tau(v_1^+) := \frac{i}{2}(H-B)(v_1^+, v_1^+) \quad \text{for } v_1 \in \Gamma_1^h \quad (6.22)$$

is known as the complex period matrix.

Proof of the theorem. The proof is a straightforward calculation. First we represent a closed $(2\ell+1)$ -form with integral periods λ as $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in \bar{V}_1$ and $\lambda_2 \in V_2 \cap \Omega_{\mathbb{Z}}^{2\ell+1}$. This decomposition is *not* unique. Any two decompositions are related by adding an exact form from V_2 . As one will see the result of the calculation does not depend on a particular choice of such a decomposition. We shift the integration variable $R \mapsto R + \lambda_1$:

$$\begin{aligned} \vartheta^\eta(a^+ + \lambda^+) = \exp\left[-\frac{\pi}{2}(H-B)(\eta^+, \eta^+) + \frac{\pi}{2}B(a^+ + \lambda^+, a^+ + \lambda^+) - \pi(H-B)(a^+ + \lambda^+, \eta^+)\right] \\ \times \int_{\bar{V}_1/V_{12}} \mathcal{D}R \exp\left[-\frac{\pi}{2}(H-B)(R^+, R^+) - \frac{\pi}{2}(H-B)(\lambda_1^+, \lambda_1^+) - \pi(H-B)(\lambda^+, R^+) \right. \\ \left. + \pi(H-B)(a^+ + \eta^+, R^+ + \lambda_1^+) + \pi(H-B)(\lambda^+, R^+ + \lambda_1^+)\right]. \end{aligned}$$

To obtain this expression we used the fact that $(H - B)$ restricted to a lagrangian subspace V_1^+ is symmetric. One sees that the terms containing R^+ and λ^+ cancel. So we can rewrite this expression as

$$\begin{aligned} \vartheta^n(a^+ + \lambda^+) &= \vartheta^n(a^+) \exp \left[\frac{\pi}{2} H(\lambda^+, \lambda^+) + \pi H(a^+, \lambda^+) \right] \\ &\quad \times \exp \left[\frac{\pi}{2} (H - B)(\lambda_2^+, \lambda^+) - \pi (H - B)(\lambda^+, \eta^+) + \pi (H - B)(\eta^+, \lambda^+) \right]. \end{aligned}$$

To obtain this expression we used the fact that $(H - B)|_{V^+ \times V_2^+} = 0$. Now since B is symmetric it cancels in the last two terms in the second line. Using the formula $H(u^+, v^+) = g(u, v) + i\omega(u, v)$ and (6.15) one finds that (6.20) satisfies the functional equation (6.21). \square

Corollary 6.1. *The partition function $\mathcal{Z}_{1,1}(a^+, a^-; \Sigma)$ is*

$$\mathcal{Z}^\varepsilon(a^+, a^-; \Sigma) = e^{i\pi\omega(a^-, a^+)} \vartheta^{\varepsilon + \sigma(\check{A}_\bullet, \Sigma)}(a^+) \quad (6.23)$$

where $\varepsilon \in \Omega_d^{2\ell+1}(X)$ is defined in (6.19).

Remark 6.2. *Note that $\sigma(\check{A}_\bullet, \Sigma)$ is a singular differential form, so the term in (6.20) which is quadratic in η requires regularization.*

Theorem 6.2. *The functional integral (6.20) for $\eta, a \in \mathcal{H}^{2\ell+1}(X)$ is equal to $\mathcal{N}_g \vartheta[\frac{\eta_1}{\eta_2}](a^+)$ where \mathcal{N}_g is an important metric dependent factor coming from the integration over the exact forms in (6.20), and $\vartheta[\frac{\eta_1}{\eta_2}](a^+)$ is the canonical theta function on the finite dimensional torus $\mathcal{H}^{2\ell+1}/\mathcal{H}_{\mathbb{Z}}^{2\ell+1}$*

$$\begin{aligned} \vartheta[\frac{\eta_1}{\eta_2}](a^+) &= \exp \left[-\frac{\pi}{2} (H - B)(\eta_2^+, \eta_1^+) + \frac{\pi}{2} B(a^+, a^+) - \pi (H - B)(a^+, \eta^+) \right] \\ &\quad \times \sum_{R \in \Gamma_1^h - \eta_1} \exp \left[-\frac{\pi}{2} (H - B)(R^+, R^+) + \pi (H - B)(a^+ + \eta_2^+, R^+) \right]. \end{aligned} \quad (6.24)$$

Here $\eta = \eta_1 + \eta_2$ according to the Lagrangian decomposition of the space of harmonic forms $\Gamma^h = \Gamma_1^h + \Gamma_2^h$.

Quantum equation of motion. The infinitesimal version of the Gauss law (5.31) for $R = dc$ yields the differential equation on \mathcal{Z}

$$d(D_a \mathcal{Z}_{1,1})(a; \Sigma) = -2\pi i \left[F(\check{A}) - \delta(\Sigma) \right] \mathcal{Z}_{1,1}(a; \Sigma). \quad (6.25)$$

Now we can apply this equation to the partition function (6.23) restricted to the real slice in the space of complexified gauge fields: $(a^+)^* = a^-$. Taking into account that \mathcal{Z} is a holomorphic section, $D^- \mathcal{Z} = 0$ we obtain the following

Theorem 6.3 (Quantum equation of motion). *The infinitesimal Gauss law yields the quantum equation of motion*

$$\langle d\mathcal{F}^-(R - \varepsilon - \sigma) \rangle_{\check{A}, \Sigma} = \delta(\Sigma) - F(\check{A}) \quad (6.26)$$

where $\mathcal{F}^-(a) := a_2^\perp - i *_E a_2^\perp$ for any $a \in \Omega^{2\ell+1}(X)$, $\sigma = \sigma(\check{A}; \Sigma)$ is defined in (5.28). $\langle \mathcal{O}(R) \rangle_{\check{A}, \Sigma}$ is the normalized correlation function defined as the ratio of the Euclidean functional integral (6.20) with the insertion of $\mathcal{O}(R)$ and the same integral without the insertion.

Proof. From (6.14a) it follows that $(H - B)(a^+, b^+) = 2i\omega(a, \mathcal{F}^-(b))$. A straightforward calculation yields

$$\frac{1}{2\pi i \mathcal{Z}^\varepsilon} D^+ \mathcal{Z}^\varepsilon(a^+, a^-; \Sigma) = \omega(\delta a, \langle \mathcal{F}^-(R - \varepsilon - \sigma) \rangle_{\check{A}, \Sigma}) + \omega(\delta a^+, (a^+)^* - a^-).$$

Now if we restrict it to the real slice $(a^+)^* = a^-$ the last term disappears. To obtain (6.26) one needs to substitute $\delta a = d\alpha$ into the equation above, integrate by parts and compare with (6.25). \square

Remark 6.3. *Suppose that we have chosen another complementary Lagrangian subspace Γ_1^h so that $\bar{\Gamma}_1^h \oplus \bar{\Gamma}_2^h$ is a decomposition for the lattice $\bar{\Gamma}^h$. The Lagrangian decompositions $\Gamma_1' \oplus \Gamma_2$ and $\Gamma_1 \oplus \Gamma_2$ are related by a linear transformation $f : \Gamma_1^h \rightarrow \Gamma_2^h$ satisfying*

$$\omega(f(v_1), u_1) + \omega(v_1, f(u_1)) = 0. \quad (6.27)$$

The function $\omega(\bar{f}(\lambda_1), \lambda_1) \bmod 2$ is a linear function on the lattice $\bar{\Gamma}_1^h$. Thus there exists a vector w_f representing the equivalence class $[w_f] \in \bar{\Gamma}_2^h / 2\bar{\Gamma}_2^h$ such that

$$\omega(w_f, \lambda_1) = \omega(f(\lambda_1), \lambda_1) \bmod 2. \quad (6.28)$$

The theta functions (6.24) corresponding to lagrangian decompositions are related by

$$\vartheta_{\Gamma_1' \oplus \Gamma_2} \left[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon_2 \end{smallmatrix} \right] (a^+) = e^{i\pi\omega(w_f, \varepsilon_1)} \vartheta_{\Gamma_1 \oplus \Gamma_2} \left[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon_2 + \frac{1}{2}w_f \end{smallmatrix} \right] (a^+). \quad (6.29)$$

The partition function in a simple case. In this paragraph we describe the properties of the partition function of a pure self-dual field, meaning that it couples neither to external gauge field nor to sources. To motivate the construction consider first the chiral scalar in two dimensions. Recall that the chiral scalar is isomorphic to a chiral Weyl fermion. It is a section of a spin bundle S_σ . To define the spin bundle S_σ globally one needs to choose a spin structure, σ . The spin structure is usually not unique. The space of spin structures is an affine space with translation group $H^1(X; \mathbb{Z}/2)$. One can think of this group as the space of isomorphism classes of topologically

trivial line bundles \mathcal{L}_α with flat connection specified by $\alpha \in H^1(X; \mathbb{Z}/2)$. The group acts by $S_\sigma \mapsto S_{\sigma+\alpha} := S_\sigma \otimes \mathcal{L}_\alpha$.

Now we choose a spin structure and couple the fermion to the external $U(1)$ gauge field A . This means that we are given a trivial line bundle \mathcal{L} with connection, and the fermion is a section of $S_\sigma \otimes \mathcal{L}$. We want now to turn off the gauge field. The surprising fact is that there are several ways to do it: one can either put the gauge field to zero and get back the fermion in S_σ , or one can put A to be a *flat* connection specified by $\alpha \in H^1(X; \mathbb{Z}/2)$, in this case one gets a fermion living in $S_{\sigma+\alpha}$.

Let us go back to our initial problem. We are given a partition function of the self-dual field coupled to an external $U(1)$ gauge field $[\check{A}]$. We would like now to turn it off. There are several ways to do it corresponding to different choices of QRIF with values in $\{\pm 1\}$.

So we assume that the characteristic class μ of the QRIF vanishes and there are no Wilson surfaces. In this case the tadpole constraint (5.27) says that \check{A} is topologically trivial, $[\check{A}] = [A]$. For our purpose it is enough to consider only flat gauge fields, so $dA = 0$. In addition we put $a = A - A_\bullet$ to zero. In this case the Gauss law (5.31) takes the form

$$\mathcal{Z}_{1,1}(R) = \Omega_0([R]) e^{-2\pi i \omega(R, A_\bullet)} \mathcal{Z}_{1,1}(0)$$

for any $R \in \Omega_{\mathbb{Z}}^{2\ell+1}(X)$. The only parameter we are left to play with is the holonomy of the base point: $[A_\bullet] \in \Omega_d^{2\ell+1}(X)/\Omega_{\mathbb{Z}}^{2\ell+1}(X)$. To fix it we require that the quadratic refinement

$$\Omega(R) := \Omega_0(R) e^{-2\pi i \omega(R, A_\bullet)}$$

take values in $\{\pm 1\}$. The space of such QRIF's can be parameterized by $[\varepsilon] \in \Omega_{\frac{1}{2}\mathbb{Z}}^{2\ell+1}/\Omega_{\mathbb{Z}}^{2\ell+1}$:

$$\Omega_\varepsilon(R) := \Omega_{\Gamma_1 \oplus \Gamma_2}(R) e^{2\pi i \omega(\varepsilon, R)}. \quad (6.30)$$

From Corollary 6.1 and Theorem 6.1 we obtain the following

Theorem 6.4. *The set of partition functions for a pure self-dual field forms a torsor for $\Omega_{\frac{1}{2}\mathbb{Z}}^{2\ell+1}/\Omega_{\mathbb{Z}}^{2\ell+1}$. The basis can be chosen as*

$$\mathcal{Z} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = e^{-i\pi \omega(\varepsilon_1, \varepsilon_2)} \int_{\bar{V}_1(\varepsilon_1)/V_{12}} \mathcal{D}R \exp \left[-\frac{\pi}{2} (H - B)(R^+, R^+) + 2\pi i \omega(\varepsilon_2, R) \right] \quad (6.31)$$

where $[\varepsilon_1] \in \frac{1}{2}\bar{\Gamma}_1/\bar{\Gamma}_1$, $[\varepsilon_2] \in \frac{1}{2}\bar{\Gamma}_2/\bar{\Gamma}_2$, we integrate over all forms in $\bar{V}_{12}(\varepsilon_1)$ modulo exact forms in V_2 and

$$\bar{V}_{12}(\varepsilon_1) = \{R \in \Omega_{\mathbb{Z}}^{2\ell+1}(X) \mid [R + \varepsilon_1]_{DR} \in \bar{\Gamma}_1\}. \quad (6.32)$$

Remark 6.4. *From (6.31) one obtains that ε_1 has interpretation of the half-integral shift in the flux quantization condition while ε_2 is the topological theta angle.*

6.3 Normalization

In this section we discuss some properties of the metric dependent normalization factor \mathcal{N}_g which appeared in Theorem 6.2. We present a calculation which fixes $\|\mathcal{N}_g\|^2$.

The partition function $\mathcal{Z}(a; \Sigma)$ restricted to $P_{q\Sigma} \cong \Omega_d^{2\ell+1}/\Omega_{\mathbb{Z}}^{2\ell+1}$ defines an element of the finite dimensional Hilbert space \mathcal{H}^{qu} . One can normalize the wave function, $\|\mathcal{Z}\|^2 = 1$, with respect to the inner product in \mathcal{H}^{qu} and in this way fix the norm square of \mathcal{N}_g .

It is clear that \mathcal{N}_g does not depend on the source Σ . So to simplify the calculation we put it zero, and assume that the characteristic class of the QRIF $\mu = 0$. In this case we can choose the base point $\check{A}_\bullet = 0$. This means that $\sigma(\check{A}_\bullet) = 0$ and $a = \check{A} - \check{A}_\bullet$ is a closed form. In Theorem 6.2 we introduced a normalization factor \mathcal{N}_g . This must be regarded as a section of a Hermitian line bundle \mathcal{L} over the space of metrics. The norm on the Hilbert space \mathcal{H}^{qu} is just the L^2 -norm on $\mathcal{L} \otimes \mathcal{L}$:

$$\|\mathcal{Z}\|_{L^2}^2 := \int_{\Omega_d^{2\ell+1}/\Omega_{\mathbb{Z}}^{2\ell+1}} \mathcal{D}a \|\mathcal{Z}(a)\|^2 \quad (6.33)$$

where the second set of $\|\cdot\|^2$ denotes the norm on \mathcal{L} .

From Theorem 6.2 and Corollary 6.1 we learn that the partition function restricted to the real slice $a^- = (a^+)^*$ can be written as

$$\mathcal{Z} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (a) = \mathcal{N}_g e^{-i\pi\omega(\varepsilon_2, \varepsilon_1)} \sum_{R \in \Gamma_1^h - \varepsilon_1} e^{-\frac{\pi}{2}(H-B)(R^+ - a_1^+, R^+ - a_1^+) + 2\pi i\omega(a_2 + \varepsilon_2, R) + i\pi\omega(a_1, a_2)} \quad (6.34)$$

where a is a *harmonic* form, $a = a_1 + a_2$ according to the Lagrangian decomposition $\Gamma^h = \Gamma_1^h \oplus \Gamma_2^h$.

To calculate the norm (6.33) we need to fix a gauge in this functional integral. This can be done by using equation (C.11). By evaluating the Gaussian integral and solving the equation $\|\mathcal{Z}\|_{L^2}^2 = 1$ for \mathcal{N}_g one finds

$$\|\mathcal{N}_g\|^2 = \left[\frac{1}{\det(\text{Im } \tau)} \prod_{p=0}^{2\ell} [V_p^{-2} \det'(L^2 d^\dagger d|_{\Omega^p(X) \cap \text{im } d^\dagger})]^{(-1)^p} \right]^{-1/2} \quad (6.35)$$

where τ is the complex period matrix defined by (6.22) (and in appendix E), V_p is the volume of the torus of harmonic p -forms $\mathcal{H}^p/\mathcal{H}_{\mathbb{Z}}^p$ defined by (C.9).

From equation (6.35) it follows that \mathcal{N}_g is some kind of square root of the right hand side of (6.35). We now conjecture that there is a very natural squareroot provided we view \mathcal{N}_g as a section of some determinant line bundle. We expect that we should set

$$\det(\text{Im } \tau)^{-1} \prod_{p=0}^{2\ell} [V_p^{-2} \det'(L^2 d^\dagger d|_{\Omega^p(X) \cap \text{im } d^\dagger})]^{(-1)^p} = \|\det \overline{\mathcal{D}}\|_Q^2 \quad (6.36)$$

where the right hand side is the Quillen norm of a section $\det \overline{\mathcal{D}}$ of some determinant line bundle $\text{DET}(\overline{\mathcal{D}})$ over the space of metrics on X . In the case of the chiral scalar $\det \overline{\mathcal{D}} = \det \bar{\partial}$. Here one can check that indeed on a two dimensional torus with metric $ds^2 = \frac{1}{\tau_2} |d\sigma^1 - \tau d\sigma^2|^2$ the integral over exact forms in (6.20) equals $1/\eta(\tau)$. This is consistent with (6.36) for $\ell = 0$. So we conjecture that this will continue to be true for $\ell > 0$.

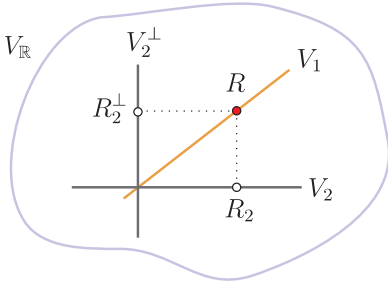
7 Action and equations of motion

The action for the self-dual field is essentially the complex period matrix (6.22) extended from the cohomology to the space of closed forms. The purpose of this section is to describe this extension.

7.1 Classical action

First we need to extend the definition of the complex period matrix (6.22) defined on the cohomology to the infinite dimensional symplectic vector space $V_{\mathbb{R}} = \Omega^{2\ell+1}$.

$(X, g_E) = \mathbf{Riemannian\ manifold}$. Following the discussion in section 6.2 we choose an orthogonal coordinate system on $V_{\mathbb{R}}$ to be $V_2 \oplus V_2^\perp$ where V_2 is a Lagrangian subspace and $V_2^\perp = *_E V_2$ is its orthogonal complement with respect to the Hodge metric. From the positivity of g_E it follows that $V_2 \cap V_2^\perp = \{0\}$. Recall that the Hodge complex structure is compatible with the symplectic



structure, thus $V_{\mathbb{R}} = V_2 \oplus V_2^\perp$ is a Lagrangian decomposition. So any form $v \in V_{\mathbb{R}}$ can be uniquely written in the form $v = v_2 + v_2^\perp$ for some $v_2 \in V_2$ and $v_2^\perp \in V_2^\perp$.

Let V_1 be another Lagrangian subspace. A choice of Lagrangian decomposition $\Gamma = \Gamma_1 \oplus \Gamma_2$ of the cohomology $\Gamma = H_{DR}^{2\ell+1}(X)$ defines a *canonical* choice of V_1 . However we postpone this discussion till the next paragraph. Now any element R from the Lagrangian subspace V_1 can be written as

$$R = R_2 + R_2^\perp \quad (7.1)$$

where R_2 and R_2^\perp are not independent but related by some linear function (see the figure). From (6.14b) it follows that the Euclidean action is

$$S_E(R^+) := \pi \int_X (R_2^\perp \wedge *_E R_2^\perp - i R_2 \wedge R_2^\perp). \quad (7.2)$$

$(M, g) = \mathbf{Lorentzian\ manifold}$. The action in the Lorentzian signature can be obtained from (7.2) by Wick rotation:

$$S_L(R) := \pi \int_M (R_2^\perp \wedge *R_2^\perp + R_2 \wedge R_2^\perp). \quad (7.3)$$

This action depends on a choice of Lagrangian subspace V_2 . For a Riemannian manifold a choice of V_2 automatically defines the Lagrangian decomposition $V_{\mathbb{R}} = V_2 \oplus *E V_2$. For a Lorentzian manifold this is not true, and we need *to constrain* the choice of V_2 by the requirement

$$V_2 \cap *V_2 = \{0\}. \quad (7.4)$$

In principle, V_2 can be an arbitrary Lagrangian subspace satisfying the constraint (7.4). Recall that on any Lorentzian manifold M there exists a nowhere vanishing timelike vector field ξ . It can be used to define a Lagrangian subspace $V_2 \subset \Omega^{2\ell+1}(M)$:

$$V_2(\xi) := \{\omega \in \Omega^{2\ell+1}(M) \mid i_\xi \omega = 0\}.$$

A choice of Lagrangian subspace V_1 . The symplectic form ω on $V_{\mathbb{R}}$ defines a symplectic form on the DeRham cohomology $\Gamma = H_{DR}^{2\ell+1}(M)$. In turn, a choice of Lagrangian subspace $V_2 \subset V_{\mathbb{R}}$ defines a Lagrangian subspace $\Gamma_2 \subset \Gamma$. We choose Γ_1 to be an arbitrary complementary Lagrangian subspace, so $\Gamma = \Gamma_1 \oplus \Gamma_2$. Choose a subspace $V_1 \subset V_{\mathbb{R}}$ to consist of all closed $2\ell+1$ -forms whose DeRham cohomology class lies in Γ_1 :

$$V_1 = \{R \in \Omega_d^{2\ell+1}(M) \mid [R]_{DR} \in \Gamma_1\}. \quad (7.5)$$

V_1 defined by (7.5) is a Lagrangian subspace of $\Omega^{2\ell+1}(M)$ (see Lemma 6.1 for proof).

Equations of motion. The variational problem for the action (7.3) is summarized by the following theorem:

Theorem 7.1. *Let $V_1 \subset V_{\mathbb{R}}$ be a Lagrangian subspace defined by (7.5), and let $R \in V_1$ be a closed form. Then the action*

$$S_L(R) = \pi \int_M (R_2^\perp \wedge *R_2^\perp + R_2 \wedge R_2^\perp) \quad (7.6)$$

has the following properties:

- a. *Variation with respect to $R \mapsto R + d\delta c$ where $\delta c \in \Omega_{cpt}^{2\ell}(M)$ is*

$$\delta S_L(R) = -2\pi \int_M \delta c \wedge d(*R_2^\perp - R_2). \quad (7.7a)$$

b. Stationary points of the action are the solutions of equations:

$$\text{Bianchi identity: } d(R_2 + R_2^\perp) = 0 \quad (7.7b)$$

$$\text{equation of motion: } d(*R_2^\perp - R_2) = 0. \quad (7.7c)$$

c. From (7.7b) and (7.7c) it follows that the following anti-self dual form is closed:

$$\mathcal{F}^+(R) := R_2^\perp + *R_2^\perp, \quad d\mathcal{F}^+(R) = 0. \quad (7.7d)$$

d. The variation (7.7a) can be written as

$$\delta S_L(R) = -2\pi \int_M \delta c \wedge d\mathcal{F}^+(R). \quad (7.7e)$$

Proof. The variation of the action is

$$\delta S_L = \pi \int_M (2\delta R_2^\perp \wedge *R_2^\perp + \delta R_2 \wedge R_2^\perp + R_2 \wedge \delta R_2^\perp).$$

The variations δR_2 and δR_2^\perp are not independent but come from the variation δR . R is an element of Lagrangian subspace and thus so must be $R + \delta R$:

$$0 = \omega(\delta R, R) = \int_M (\delta R_2 \wedge R_2^\perp + \delta R_2^\perp \wedge R_2). \quad (7.8)$$

Using this constraint one easily finds that the variation of the action is

$$\delta S_L = 2\pi \int_M \delta R_2^\perp \wedge \underbrace{(*R_2^\perp - R_2)}_{\in V_2}.$$

Notice that the expression in the brackets is in the Lagrangian subspace V_2 . Thus we can substitute δR instead of δR_2^\perp :

$$\delta S_L = 2\pi \int_M \delta R \wedge (*R_2^\perp - R_2).$$

Recall that we must vary R within a fixed cohomology class, so $\delta R = d\delta c$ where $\delta c \in \Omega_{cpt}^{2\ell}(M)$:

$$\delta S_L = -2\pi \int_M \delta c \wedge d(*R_2^\perp - R_2) \stackrel{dR=0}{=} -2\pi \int_M \delta c \wedge d(\underbrace{R_2^\perp + *R_2^\perp}_{=: \mathcal{F}^+}) = -2\pi \int_M \delta c \wedge d\mathcal{F}^+. \quad (7.9)$$

Note that $\mathcal{F}^+(R)$ is *automatically* self dual. Since δc is an arbitrary 2ℓ -form it immediately follows that $d\mathcal{F}^+ = 0$. \square

Remark 7.1. *The action for anti self-dual field can be obtained from (7.6) by changing the sign of the second term.*

Theorem 7.2. *An arbitrary **closed** self dual form \mathcal{F}^+ can be written in form $\mathcal{F}^+(R)$ for some $R \in V_1$.*

Proof. A closed self-dual form defines the self-dual DeRham cohomology class $[\mathcal{F}^+]_{DR} \in H_{DR}^{2\ell+1}(M)$. If this DeRham cohomology class is zero then \mathcal{F}^+ is an exact form, and we can take $R = \mathcal{F}^+ \in V_1$.

If $[\mathcal{F}^+]_{DR} \neq 0$ then we choose a self-dual representative α^+ of this cohomology class. Then $\mathcal{F}^+ - \alpha^+$ is the self-dual *exact* form. From the constraint $V_2 \cap *V_2 = \{0\}$ it follows that $\Gamma_2 \cap [* \Gamma_2] = [0]$ in the DeRham cohomology. Thus the class α^+ can be written as $\alpha_2 + *\alpha_2$ for some representative α_2 of the cohomology class $[\alpha_2] \in \Gamma_2$. From the fact that Γ_1 and Γ_2 are complementary Lagrangian subspaces it follows that the class $[\alpha_2]$ can be obtained by the orthogonal projection from some class $[\alpha_1] \in \Gamma_1$. So choose an arbitrary representative α_1 of the class $[\alpha_1]$ and consider $R = \alpha_1 + (\mathcal{F}^+ - \alpha^+)$. By the construction R is in V_1 and $\mathcal{F}^+(R) = \mathcal{F}^+$. \square

Gauge symmetries. The action (7.6) in addition to the standard gauge symmetry $C \mapsto C + \omega$ where ω is closed 2ℓ -form with integral periods has an extra gauge symmetry. Indeed, the functional $S_L(R)$ vanishes on the Lagrangian subspace V_2 . The Lagrangian subspaces V_1 and V_2 have non-zero intersection which we denote by

$$V_{12} := V_1 \cap V_2 = \{\text{exact forms in } V_2\}. \quad (7.10)$$

For any $R \in V_1$ and $v_{12} \in V_{12}^{cpt}$ (compactly supported forms in V_{12}) it follows that

$$S_L(R + v_{12}) = S_L(R) + S_L(v_{12}) = S_L(R). \quad (7.11)$$

This properties can be summarized by the following theorem:

Theorem 7.3. *The action (7.6) has two types of gauge symmetries:*

- a. *It manifestly invariant with respect $C \mapsto C + \omega$ where $\omega \in \Omega_{\mathbb{Z}}^{2\ell}(M)$.*
- b. *It is invariant under a shift $R \mapsto R + v_{12}$ where $v_{12} \in (V_1 \cap V_2)_{cpt}$:*

$$S_L(R + v_{12}) = S_L(R). \quad (7.12)$$

The anti self-dual field (7.7d) does not depend on v_{12} :

$$\mathcal{F}^+(R + v_{12}) = \mathcal{F}^+(R). \quad (7.13)$$

From this theorem it follows that the gauge symmetry $R \mapsto R + v_{12}$ does not affect classical equations. However this extra gauge symmetry has to be taken into account in the quantum theory.

Coupling to the sources. One can use Theorem 6.1 to write a coupling of the self-dual field to a brane.

Theorem 7.4. *Let Σ be a topologically trivial 2ℓ -cycle on M (it might have several connected components). The differential character $[\check{\delta}(\Sigma)]$ is topologically trivial and is represented by a (singular) $(2\ell + 1)$ -form $\sigma \in [\sigma] \in \Omega^{2\ell+1}(M)/\Omega_{\mathbb{Z}}^{2\ell+1}(M)$: $d\sigma = \delta(\Sigma)$. The action for the self-dual field coupled to the brane is*

$$S_L(R) := \pi \int_M (R_2^\perp \wedge *R_2^\perp + R_2 \wedge R_2^\perp) + 2\pi \int_M R_2^\perp \wedge (*\sigma_2^\perp - \sigma_2). \quad (7.14)$$

The variation of the action with respect to $R \mapsto R + d\delta c$ for $\delta c \in \Omega_{cpt}^{2\ell}(M)$ is

$$\delta S_L(R) = 2\pi \int_M \delta c \wedge [\delta(\Sigma) - d\mathcal{F}^+(R + \sigma)]. \quad (7.15)$$

Remark 7.2. *The equations of motion can be written as*

$$dR = 0, \quad d\mathcal{F}^+(R + \sigma) = \delta(\Sigma) \quad \text{for } R \in V_1 \quad \text{and} \quad [\sigma] = [\check{\delta}(\Sigma)]. \quad (7.16)$$

Remark 7.3. *The condition $[\sigma] = [\check{\delta}(\Sigma)]$ fixes only an equivalence class $[\sigma] \in \Omega^{2\ell+1}(M)/\Omega_{\mathbb{Z}}^{2\ell+1}(M)$ but not σ itself. The action (7.14) depends on a choice of a representative σ of $[\sigma]$. However the partition function (6.20) does not depend on such a choice.*

Proof. The action (7.14) can be directly obtained from Theorem 6.1 and Corollary 6.1 by setting $\check{A} = 0$. Using the result of Theorem 7.1 one finds

$$\delta S_L(R) = -2\pi \int_M \delta c \wedge d\mathcal{F}^+(R) + 2\pi \int_M \delta R_2^\perp \wedge (*\sigma_2^\perp - \sigma_2).$$

The term in the brackets lies in the Lagrangian subspace V_2 thus we can change δR_2^\perp to $\delta R = d\delta c$. Integrating by parts and using that $d\sigma = \delta(\Sigma)$ and $\mathcal{F}^+(R) := R_2^\perp + *R_2^\perp$ one arrives at equation (7.15). \square

7.2 Examples

In this section we consider two examples: a chiral scalar field on $\mathbb{R}^{1,1}$ and a self-dual field on a production manifold $M = \mathbb{R} \times N$ where N is compact $4\ell + 1$ -manifold.

7.2.1 Chiral scalar on \mathbb{R}^2 .

Consider a Lorentzian manifold $M = \mathbb{R}^2$ equipped with the metric $ds^2 = e^{2\varphi(x,t)}(-dt^2 + dx^2)$. Choose the Lagrangian subspace V_2 as

$$V_2 = \{dt \omega_t(x, t)\} \quad \Rightarrow \quad V_2^\perp = \{dx \omega_x(x, t)\}. \quad (7.17)$$

The Lagrangian subspace V_1 is just the space of all exact 1-forms $\Omega_{\text{exact}}^1(M)$. So $R \in V_1$ decomposes as

$$R = \underbrace{dx R_x}_{R_2^\perp} + \underbrace{dt R_t}_{R_2}. \quad (7.18)$$

The action (7.6) takes the form

$$S_L(R) = \pi \int_{\mathbb{R}^2} dt dx R_x (R_x + R_t).$$

Now for $R = d\phi$ the action becomes

$$S_L(\phi) = \pi \int_{\mathbb{R}^2} dt dx \left[(\partial_x \phi)^2 + (\partial_x \phi)(\partial_t \phi) \right]. \quad (7.19)$$

This action for the chiral scalar has been proposed before [5]. The equation of motion is

$$(\partial_x + \partial_t)\partial_x \phi = 0. \quad (7.20)$$

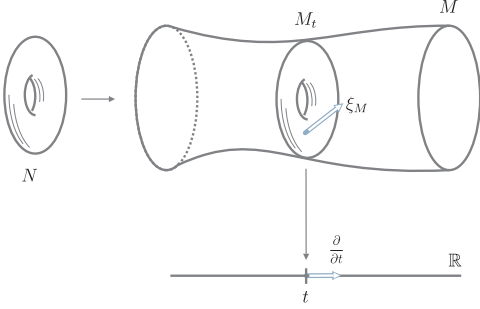
Thus the general solution is $\phi(x, t) = f(t) + \phi_L(x - t)$. It seems that we get an extra degree of freedom represented by an arbitrary function of time $f(t)$. However the anti self-dual field \mathcal{F}^+ depends only on $\phi_L(x - t)$. Indeed, substituting the solution to equation (7.7d) one finds

$$\mathcal{F}^+(\phi) = (dx - dt)\phi'_L(x - t) \quad (7.21)$$

where ϕ'_L denotes the derivative of ϕ_L with respect to the argument.

The extra degree of freedom $f(t)$ is the gauge degree of freedom (7.12), and it can be removed by the gauge transformation $R \mapsto R - df(t)$ where $-df(t) \in V_1 \cap V_2$.

7.2.2 Action for self-dual field on a product manifold



Not every $(4\ell+2)$ -manifold M admits a smooth Lorentzian metric¹³. However if M is a product manifold say $\mathbb{R} \times N$ then there exists a nowhere vanishing vector field, and thus M admits a Lorentzian metric. It is convenient to think of $M = \mathbb{R} \times N$ as a topologically trivial fibration M over \mathbb{R} with fiber N .

On \mathbb{R} there is a canonical nowhere vanishing vector field $\partial/\partial t$. To lift it to M we need to choose a connection on M . A connection Θ is a globally well defined 1-form on M with values in the Lie algebra of $\text{Diff}^+(N)$, i.e. $\Theta \in \Omega^1(M; \text{Vect}(N))$. A choice of Riemannian metric g_N on N and Lorentzian metric $ds_{\mathbb{R}}^2 = -\rho^2 dt^2$ on \mathbb{R} defines a Lorentzian metric on M

$$ds_M^2 = -\rho^2 dt^2 + g_N(\Theta, \Theta). \quad (7.22a)$$

In local coordinates (t, x^i) the connection Θ can be written as

$$\Theta = (dx^i - \xi^i dt) \otimes \frac{\partial}{\partial x^i}, \quad (7.22b)$$

and the metric takes the familiar form

$$ds_M^2 = -\rho^2 dt^2 + (g_N)_{ij}(dx^i - \xi^i dt)(dx^j - \xi^j dt). \quad (7.22c)$$

In general relativity ρ is usually called lapse and ξ^j the shift.

The vector field $\partial/\partial t$ on \mathbb{R} lifts to a vector field ξ_M on M :

$$i_{\xi_M} \Theta = 0 \quad \Rightarrow \quad \xi_M := \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i}. \quad (7.23)$$

The connection Θ defines a decomposition of tangent bundle $T_x M$ on horizontal and vertical vector fields. This decomposition is the orthogonal decomposition in the metric (7.22a).

Decomposition of forms. An orthogonal projector onto the space of horizontal vectors is defined by

$$P(\eta) := \xi_M \frac{g(\xi_M, \eta)}{g(\xi_M, \xi_M)} = \xi_M dt(\eta).$$

¹³Recall that oriented manifold M admits a Lorentzian metric if and only if there exists a nowhere vanishing vector field.

The dual projector $P^* := dt \otimes i_{\xi_M}$ defines a decomposition of the differential forms onto vertical and horizontal:

$$\Omega^{2\ell+1}(M) = \underbrace{P^*\Omega^{2\ell+1}(M)}_{\text{horizontal}} \oplus \underbrace{(1 - P^*)\Omega^{2\ell+1}(M)}_{\text{vertical}}. \quad (7.24)$$

Since M is a product manifold there is another decomposition of

$$\Omega^{2\ell+1}(M) = \Omega^{2\ell+1}(N) \otimes \Omega^0(\mathbb{R}) \oplus \Omega^{2\ell}(N) \otimes \Omega^1(\mathbb{R}). \quad (7.25)$$

These two decomposition are related in the following way

$$R = \bar{R} + dt \wedge \bar{R}_0 = \underbrace{(\bar{R} - dt \wedge i_{\xi} \bar{R})}_{\text{vertical}} + \underbrace{dt \wedge (\bar{R}_0 + i_{\xi} \bar{R})}_{\text{horizontal}}. \quad (7.26)$$

The Hodge $*_M$ exchanges the vertical and horizontal forms

$$*_M R = \underbrace{-\rho dt \wedge *_N \bar{R}}_{\text{horizontal}} + \underbrace{\left\{ -\rho^{-1} *_N (\bar{R}_0 + i_{\xi} \bar{R}) + dt \wedge i_{\xi} [\rho^{-1} *_N (\bar{R}_0 + i_{\xi} \bar{R})] \right\}}_{\text{vertical}} \quad (7.27)$$

Lagrangian subspaces. We choose a Lagrangian subspace $V_2 = \Omega^{2\ell}(N) \otimes \Omega^1(\mathbb{R})$. Notice that it coincides with the space of horizontal $2\ell + 1$ -forms, thus $*_M V_2 = \{\text{vertical forms}\}$.

The DeRham cohomology decomposes as

$$H_{DR}^{2\ell+1}(M) \cong H_{DR}^{2\ell+1}(N) \oplus H_{DR}^{2\ell}(N). \quad (7.28)$$

A choice of Lagrangian subspace V_2 yields a choice for Γ_2 : $\Gamma_2 \cong H^{2\ell}(N)$. Choose $\Gamma_1 \cong H^{2\ell+1}(N)$. Now the Lagrangian subspace V_1 is defined by

$$V_1 = \{R \in \Omega_d^{2\ell+1}(M) \mid [R]_{DR} \in H_{DR}^{2\ell+1}(N)\}. \quad (7.29)$$

Action. Given a form $R \in V_1$ we can write it as in (7.26)

$$R = \underbrace{(\bar{R} - dt \wedge i_{\xi} \bar{R})}_{R_2^\perp} + \underbrace{dt \wedge (\bar{R}_0 + i_{\xi} \bar{R})}_{R_2}. \quad (7.30)$$

Now substituting this decomposition into the action (7.3) and using (7.27) one finds

$$S_L(R) = \pi \int_{\mathbb{R}} dt \int_N [\rho \bar{R} \wedge *_N \bar{R} + (\bar{R}_0 + i_{\xi} \bar{R}) \wedge \bar{R}]. \quad (7.31)$$

Gauge symmetry. The action (7.31) has an extra gauge symmetry (7.12):

$$V_{12} = V_1 \cap V_2 = \{dt \wedge (\dot{\bar{c}} - d_N \bar{c}_0) \mid \bar{c} \in \Omega_{d_N}^{2\ell}(N) \otimes \Omega^0(\mathbb{R}) \text{ and } \bar{c}_0 \in \Omega^{2\ell-1}(N) \otimes \Omega^1(\mathbb{R})\}. \quad (7.32)$$

Equations of motion. The equation of motion (7.7c) take the form

$$d_N[\rho *_N \bar{R} + \bar{R}_0 + i_\xi \bar{R}] = 0, \text{ or using Bianchi identity } \dot{\bar{R}} = d_N(\rho *_N \bar{R} - i_\xi \bar{R}). \quad (7.33)$$

Let $\bar{\alpha}$ be a time-independent $2\ell + 1$ -form on N representing the DeRham cohomology class of $[R]_{DR} \in \Gamma_1$. R can be written as

$$R = \alpha + d_N \bar{c} + dt \wedge (\dot{\bar{c}} - d_N \bar{c}_0).$$

Substituting this to (7.33) one finds

$$d_N[\dot{\bar{c}} - \rho *_N (\bar{\alpha} + d_N \bar{c}) + i_\xi (\bar{\alpha} + d_N \bar{c})] = 0. \quad (7.34)$$

Notice that arbitrary (t -dependent) closed 2ℓ -form solves this equation. This set of solution corresponds to the gauge degrees of freedom containing in (7.32). One should not worry about this set of solutions because they are projected out from the expression for the self-dual field:

$$\mathcal{F}^+ = \bar{R} - dt \wedge [\rho *_N \bar{R} - i_\xi \bar{R}]|_{\bar{R}=\bar{\alpha}+d_N \bar{c}}. \quad (7.35)$$

7.3 Comparison with Henneaux-Teitelboim action

In this section we compare with the previous work of Henneaux and Teitelboim. We begin by reviewing briefly their work [9].

Henneaux-Teitelboim action. Let $M = \mathbb{R} \times N$ be a product manifold equipped with the Lorentzian metric (7.22c). Consider a *closed* $2\ell + 1$ -form F on M (locally $F = dC$). It can be decomposed as in (7.26)

$$F = dt \wedge \bar{F}_0 + \bar{F}, \quad d_N \bar{F} = 0 \quad \text{and} \quad \dot{\bar{F}} = d_N \bar{F}_0. \quad (7.36)$$

The equation of motion for F is the self-duality constraint. The self-duality constraint $*_M F = F$ is equivalent to

$$\rho *_N \bar{F} + \bar{F}_0 + i_\xi \bar{F} = 0. \quad (7.37)$$

The action of Henneaux and Teitelboim [9] can be summarized by the following two theorems:

Theorem 7.5. *The action*

$$S_{HT}(F) := \int_{\mathbb{R}} dt \int_N [\rho \bar{F} \wedge *_N \bar{F} + (\bar{F}_0 + i_\xi \bar{F}) \wedge \bar{F}] \quad (7.38)$$

for the closed $2\ell + 1$ -form $F = dt \wedge \bar{F}_0 + \bar{F}$ has the following properties:

1. it is manifestly invariant under $C \mapsto C + \omega$ where $\omega \in \Omega_{\mathbb{Z}}^{2\ell+1}(M)$.
2. the variation of the action under $F \mapsto F + d_N \delta \bar{c} + dt \wedge (\delta \dot{\bar{c}} - d_N \bar{\delta} \bar{c}_0)$ is

$$\delta S_{HT} = -2 \int_{\mathbb{R}} dt \int_N \delta \bar{c} \wedge d_N [\rho *_N \bar{F} + \bar{F}_0 + i_\xi \bar{F}]. \quad (7.39)$$

Here $\delta \bar{c}$ and $\delta \bar{c}_0$ are t -dependent 2ℓ - and $(2\ell - 1)$ -forms on N with compact support.

3. it has an additional gauge symmetry $\bar{F}_0 \mapsto \bar{F}_0 + \dot{\bar{\lambda}} - d_N \bar{\lambda}_0$ where $\bar{\lambda}$ is a closed t -dependent 2ℓ -form on N with compact support, and $\bar{\lambda}_0$ is an arbitrary t -dependent $(2\ell - 1)$ -form on N .

Theorem 7.6. *A family of closed forms*

$$F_{\bar{\lambda}, \bar{\lambda}_0} = \bar{F} + dt \wedge (\bar{F}_0 + \dot{\bar{\lambda}} - d_N \bar{\lambda}_0) \quad (7.40)$$

where $\bar{\lambda} \in \Omega_{d_N}^{2\ell}(N) \otimes \Omega^0(\mathbb{R})$ and $\bar{\lambda}_0 \in \Omega^{2\ell-1}(N)$ contains a self-dual form $\mathcal{F}^+ = F_{\bar{\lambda}^*, \bar{\lambda}_0^*}$ if and only if

$$d_N [\rho *_N \bar{F} + \bar{F}_0 + i_\xi \bar{F}] = 0, \quad (7.41)$$

i.e. iff the family $F_{\bar{\lambda}, \bar{\lambda}_0}$ is an extremum of the action (7.38).

The idea of the proof is as follows [44]. If F is closed and self-dual then because of equation (7.37) it satisfies (7.41). The converse goes as follows: If (7.41) is satisfied then

$$\rho *_N \bar{F} + \bar{F}_0 + i_\xi \bar{F} = \omega_{2\ell}(t)$$

where $\omega_{2\ell}$ is a closed t -dependent 2ℓ -form on N . Using the Riemannian metric g_N on N we can decompose $\omega_{2\ell}$ on harmonic and exact parts $\omega_{2\ell} = \omega_{2\ell}^h(t) + d_N \alpha_{2\ell-1}(t)$. Now choose

$$\bar{\lambda}^*(t) := \int^t dt' \omega_{2\ell}^h(t') \quad \text{and} \quad \bar{\lambda}_0^*(t) := -\alpha_{2\ell-1}(t).$$

The form $F_{\bar{\lambda}^*, \bar{\lambda}_0^*}$ is self-dual.

Comparison of the two actions. By comparing expressions (7.38) and (7.31) one finds that modulo a change of notation they are identical. However there is an important difference: in the Henneaux-Teitelboim approach one tries to get a self-duality constraint from the variation of the action. By contrast, in our approach we get a condition that a certain self-dual form is closed.

8 Dependence on metric

Let us recall the general strategy in geometric quantization of Chern-Simons theories. The partition function is a covariantly constant section of the vector bundle $\mathcal{H}_J^{qu} \rightarrow \mathcal{T}$:

$$(\nabla_J \mathcal{Z})(A^+, A^-; J) = 0. \quad (8.1)$$

A change of polarization can be interpreted as a change of creation/annihilation operators corresponding to a Bogoliubov transformation. Thus, a change of complex structure can be compensated by a Bogoliubov transformation. The Bogoliubov transformation is implemented by a quadratic exponential, and infinitesimally it is represented by second order differential operator:

$$\nabla_J^{1,0} = \delta_J^{1,0} - (\dots) D^+ D^+ \quad \text{and} \quad \nabla_J^{0,1} = \delta_J^{0,1}.$$

In our case the complex structure is determined by the Hodge metric: $J = -*_E$. Thus eq. (2.4) implies that the metric g_X couples only to \mathcal{F}^+ , the self-dual part of \check{C} .

A trick with metric variation. In this section we study dependance of the action on the choice of metric g . In particular we need to calculate the variation $\delta_g(*\omega_k)$. In local coordinates the Hodge $*$ is defined by ¹⁴

$$*\omega_k = \frac{1}{k!(d-k)!} |\det g|^{1/2} \omega_{\mu_1 \dots \mu_k} g^{\mu_1 \nu_1} \dots g^{\mu_k \nu_k} \varepsilon_{\nu_1 \dots \nu_k \alpha_1 \dots \alpha_{d-k}} dx^{\alpha_1} \dots dx^{\alpha_{d-k}}. \quad (8.2)$$

Now it is easy to see that the variation of this expression with respect to the metric consists of two parts which can be elegantly written as (for a metric of any signature)

$$\delta_g(*)\omega_k = -\frac{1}{2} \text{tr}(\delta g^{-1} g) * \omega_k + *(\xi_g \omega_k) \quad (8.3a)$$

where

$$\xi_g := (\delta g^{-1} g)^\mu{}_\nu dx^\nu \wedge i\left(\frac{\partial}{\partial x^\mu}\right) \quad (8.3b)$$

The formula (8.3a) is the main computational tool in this section.

¹⁴We use convention in which

$$\alpha_k \wedge *\beta_k = \frac{1}{k!} \alpha_{\mu_1 \dots \mu_k} \beta_{\nu_1 \dots \nu_k} g^{\mu_1 \nu_1} \dots g^{\mu_k \nu_k} \text{vol}(g).$$

8.1 Stress-energy tensor

Let us recall a few facts about the stress-energy tensor of a $(2\ell + 1)$ -form on a $(4\ell + 2)$ -dimensional Lorentzian manifold (M, g) . The stress-energy tensor for an unconstrained $(2\ell + 1)$ -form F is

$$T_{\mu\nu}(F) := \frac{\pi}{2(2\ell + 1)!} \left[(2\ell + 1) F_{\mu\alpha_1 \dots \alpha_{2\ell}} F_{\nu}{}^{\alpha_1 \dots \alpha_{2\ell}} - \frac{1}{2} g_{\mu\nu} F_{\alpha_1 \dots \alpha_{2\ell+1}} F^{\alpha_1 \dots \alpha_{2\ell+1}} \right].$$

Working with such expressions in local coordinates proves to be cumbersome, and it is better to proceed in the following coordinate-independent way. This stress-energy tensor was obtained from the following action

$$S = \frac{\pi}{2} \int_X F \wedge *F,$$

and, using (8.3a) one can easily verify that

$$\delta g^{\mu\nu} T_{\mu\nu}(F) \text{vol}(g) := \frac{\pi}{2} [\xi_g F \wedge *F - \frac{1}{2} \text{tr}(\delta g^{-1} g) F \wedge *F]. \quad (8.4)$$

We will use this coordinate free expression as a working definition for the stress-energy tensor.

The form F can be written as a sum $F^+ + F^-$ of self-dual and anti self-dual forms: $F^\pm := \frac{1}{2}(F \pm *F)$. $T_{\mu\nu}(F^\pm)$ have the following expression in terms of F (see appendix D for derivation):

$$\delta g^{\mu\nu} T_{\mu\nu}(F^\pm) \text{vol}(g) = \frac{\pi}{2} F^\pm \wedge \xi_g F^\pm = \frac{\pi}{4} \left[\xi_g F \wedge (*F \pm F) - \frac{1}{2} \text{tr}(\delta g^{-1} g) F \wedge *F \right]. \quad (8.5)$$

From here it follows that the stress-energy tensor for $2\ell + 1$ -form factorizes as [48]

$$T_{\mu\nu}(F^+ + F^-) = T_{\mu\nu}(F^+) + T_{\mu\nu}(F^-). \quad (8.6)$$

This is the reason for the existence of a chiral splitting of the normalization function $\|\mathcal{N}_g\|^2$ in (6.35).

8.2 Stress-energy tensor for the anti self-dual field

In section 7 we derived the following action for the self-dual field on a Lorentzian manifold (M, g) :

$$S_L(R) = \pi \int_M (R_2^\perp \wedge *R_2^\perp + R_2 \wedge R_2^\perp) \quad (8.7)$$

where R is a closed form belonging to the Lagrangian subspace $V_1 \subset \Omega^{2\ell+1}(M)$ and $R = R_2 + R_2^\perp$ according to the Lagrangian decomposition $\Omega^{2\ell+1}(M) = V_2 \oplus I_L(V_2)$, $I_L := *_g$.

In this section we prove the following theorem

Lemma 8.1. *The action (7.31) is invariant under action of $\text{Diff}_0^+(M)$. The classical stress-energy tensor $T_{\mu\nu}(\mathcal{F}^+)$ is conserved for solutions of the equations of motion.*

Proof. Let η be a vector field on M with a compact support, then the variation of the action is

$$\delta_\eta S(R) = \pi \int_M (2\delta_\eta R_2^\perp \wedge *R_2^\perp + R_2^\perp \wedge \delta_\eta(*)R_2^\perp + \delta_\eta R_2 \wedge R_2^\perp + R_2 \wedge \delta_\eta R_2^\perp).$$

The variation $\delta_\eta R_2$ consists of two terms: the first comes from the variation of $R \mapsto R + L_\eta R$ and the second comes from the variation of the metric $g_{\mu\nu} \mapsto g_{\mu\nu} + (L_\eta g)_{\mu\nu}$. The first variation is just the variation (7.9) with $\delta c = i_\eta R$. The second variation is equal to (8.8) with $\delta g^{\mu\nu} = \nabla^{(\mu} \eta^{\nu)}$ where ∇_μ denotes the Levi-Civita covariant derivative. Thus we obtain

$$\delta_\eta S(R) = 2\pi \int_M i_\eta R \wedge d\mathcal{F}^+(R) + \int_M \text{vol}(g) \nabla^\mu \eta^\nu T_{\mu\nu}(\mathcal{F}^+). \quad (8.12)$$

From these equation it follows that, using the equations of motion $\nabla^\mu T_{\mu\nu}(\mathcal{F}^+) = 0$. \square

Of course, quantum mechanically there is a gravitational anomaly. Understanding this fully is part of the understanding of the factor \mathcal{N}_g .

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A Differential cocycles and cohomology

From [29] page 9.

Differential cocycle. For a manifold X the category of *differential n -cocycles*, $\check{\mathcal{H}}^n(X)$ is the category whose **objects** are triples

$$(c, h, \omega) \in C^n(X; \mathbb{Z}) \times C^{n-1}(X; \mathbb{R}) \times \Omega^n(X),$$

satisfying (the cocycle condition)

$$d(c, h, \omega) := (\delta c, \omega - c - \delta h, d\omega) = 0. \quad (\text{A.1})$$

The set of these cocycles is denoted $\check{Z}^n(X)$. Note that $d^2 = 0$. Here $C^n(X; \mathbb{Z})$ and $C^{n-1}(X; \mathbb{R})$ are n -cochains and $(n-1)$ -cochains with integral and real coefficients respectively; $\Omega^n(X)$ denotes the space of n -forms on X . A **morphism** from (c_1, h_1, ω_1) to (c_2, h_2, ω_2) is defined by an equivalence class of pairs

$$(b, k, 0) \in C^{n-1}(X; \mathbb{Z}) \times C^{n-2}(X; \mathbb{R})$$

with the action

$$(c_1, h_1, \omega_1) = (c_2, h_2, \omega_2) + d(b, k, 0). \quad (\text{A.2})$$

The equivalence relation on (b, k) is generated by

$$(b, k, 0) \sim (b, k, 0) - d(a, k', 0). \quad (\text{A.3})$$

The set of isomorphism classes of objects in the category $\check{\mathcal{H}}^n(X)$ is the Cheeger-Simons cohomology group $\check{H}^n(X)$.

Integral Wu-structures Let $p : E \rightarrow S$ be a smooth map, and fix a cocycle $\nu \in Z^{2k}(E; \mathbb{Z}/2)$ representing the Wu-class ν_{2k} of the relative normal bundle. A *differential integral Wu-structure of degree $2k$* on E/S is a differential cocycle

$$\lambda = (c, h, \omega) \in \check{Z}^{2k}(E)$$

with the property that $c \equiv \nu \pmod{2}$.

Category of shifted differential cocycles. Let M be a manifold,

$$\nu \in Z^{2k}(M; \mathbb{R}/\mathbb{Z})$$

a smooth cocycle, and $2k \geq 0$ an integer. The category of ν -twisted differential $2k$ -cocycles, $\check{\mathcal{H}}_\nu^{2k}(M)$, is the category whose objects are triples

$$(c, h, \omega) \in C^{2k}(M; \mathbb{R}) \times C^{2k-1}(M; \mathbb{R}) \times \Omega^{2k}(M),$$

satisfying

$$c \equiv \nu \pmod{\mathbb{Z}} \quad \text{and} \quad d(c, h, \omega) = 0. \quad (\text{A.4})$$

A morphism from (c_1, h_1, ω_1) to (c_2, h_2, ω_2) is an equivalence class of pairs

$$(b, k) \in C^{2k-1}(M; \mathbb{Z}) \times C^{2k-2}(M; \mathbb{R})$$

satisfying $(c_2, h_2, \omega_2) = (c_1, h_1, \omega_1) + d(b, k, 0)$. The equivalence relation is generated by

$$(b, k) \sim (b - \delta a, k + \delta k' + a).$$

B Theta function

This appendix summarizes several chapters from [42].

Line bundles over a complex torus. Let $V_{\mathbb{R}}$ denotes a real vector space of dimension $2g$ and Λ a lattice inside $V_{\mathbb{R}}$. The lattice Λ is a discrete subgroup of $V_{\mathbb{R}}$ of rank $2g$. It acts on $V_{\mathbb{R}}$ by addition. Choose a complex structure J and define a complex vector space V^+ by $V_{\mathbb{R}} \otimes \mathbb{C} = V^+ \oplus V^-$. This decomposition defines the embedding Λ^+ of the lattice Λ into V^+ . The quotient $T = V^+/\Lambda^+$ is a complex torus.

Denote the group of holomorphic line bundles on T by $\text{Pic}(T)$. Each element of the Picard group defines a holomorphic line bundle. The Picard group is described by the following exact sequence

$$1 \rightarrow \text{Pic}^0(T) \rightarrow \text{Pic}(T) \rightarrow \text{NS}(T) \rightarrow 0. \quad (\text{B.1})$$

Here $\text{NS}(T)$ is called Neron-Severi group and $\text{Pic}^0(T)$ is the connected component of 0. For $T = V^+/\Lambda^+$ one can think of it either as \mathbb{R} -linear real alternating forms on $V_{\mathbb{R}}$ satisfying $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(Jx, Jy) = E(x, y)$, or as the set of hermitian forms on V^+ with $\text{Im } H(\Lambda^+, \Lambda^+) \subseteq \mathbb{Z}$. There is a one-to-one correspondence between hermitian forms and alternating \mathbb{R} -bilinear forms on $V_{\mathbb{R}}$

satisfying $E(Jx, Jy) = E(x, y)$ for $x, y \in V_{\mathbb{R}}$. Indeed, any element u of V^+ can be *uniquely* written as $u = x^+ := \frac{1}{2}(1 - iJ)x$ for some $x \in V_{\mathbb{R}}$. Define

$$H(x^+, y^+) := E(Jx, y) + iE(x, y).$$

It is easy to verify that H defines a hermitian form on the complex vector space V^+ . In our conventions H is \mathbb{C} linear in the first argument and satisfies $H(u, v) = \overline{H(v, u)}$ for $u, v \in V^+$.

One can show that there are isomorphisms

$$\text{Pic}^0(T) \cong \text{Hom}(\Lambda, U(1)) \quad \text{and} \quad \text{Pic}(T) \cong \mathcal{P}(\Lambda) \quad (\text{B.2})$$

where $\mathcal{P}(\Lambda)$ is the set of pairs (H, χ) where $H \in \text{NS}(T)$ and χ is a semicharacter for H . A *semicharacter* for a hermitian form H is a map $\chi : \Lambda \rightarrow U(1)$ satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu) e^{-i\pi \text{Im } H(\lambda, \mu)} \quad (\text{B.3})$$

for all $\lambda, \mu \in \Lambda$. Now using the realization of the Picard group in terms of $\mathcal{P}(\Lambda)$ we can describe holomorphic line bundles over T by the factor of automorphy. Having a pair $(H, \chi) \in \mathcal{P}(\Lambda)$ we define a *canonical factor of automorphy* $a_{(H, \chi)} : \Lambda \times V \rightarrow \mathbb{C}^*$ by

$$a_{(H, \chi)}(\lambda^+, v^+) := \chi(\lambda) e^{\pi H(v^+, \lambda^+) + \frac{\pi}{2} H(\lambda^+, \lambda^+)} \quad (\text{B.4})$$

This map satisfies the cocycle relation

$$a_{(H, \chi)}(\lambda^+ + \mu^+, v^+) = a_{(H, \chi)}(\lambda^+, v^+ + \mu^+) a_{(H, \chi)}(\mu^+, v^+).$$

The cocycle defines a line bundle $L(H, \chi)$ by

$$L(H, \chi) \cong (V \times \mathbb{C})/\Lambda \quad (\text{B.5})$$

where Λ acts on $V \times \mathbb{C}$ by $\lambda \circ (v^+, t) = (v^+ + \lambda^+, a_{(H, \chi)}(\lambda^+, v^+) t)$. This means that sections of the line bundle $L(H, \chi)$ over T are those section of the trivial line bundle over V which satisfy equation

$$\vartheta(v^+ + \lambda^+) = a_{(H, \chi)}(\lambda^+, v^+) \vartheta(v^+)$$

One can prove that for every line bundle L over T there is a unique pair (H, χ) such that $L \simeq L(H, \chi)$.

Characteristics. Recall that H is a hermitian form on V whose alternating form $E = \text{Im } H$ is integral valued on the lattice Λ . The alternating form E defines a symplectic structure on the vector space V . To construct a section of the line bundle $L(H, \chi)$ we have to choose a Lagrangian decomposition of V :

$$V = V_1 \oplus V_2. \quad (\text{B.6})$$

The Lagrangian decomposition of V must be such that $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ is a Lagrangian decomposition of Λ .

Such a decomposition leads to an explicit description of all line bundles L in $\text{Pic}^H(T)$. For this we define a map $\chi_0 : V \rightarrow \mathbb{C}$ by

$$\chi_0(v) = e^{i\pi E(v_1, v_2)} \quad (\text{B.7})$$

where $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. The map χ_0 satisfies equation

$$\chi_0(u + v) = \chi_0(u)\chi_0(v) e^{i\pi E(u, v)} e^{-2\pi i E(u_2, v_1)}. \quad (\text{B.8})$$

Thus $\chi_0|_\Lambda$ is a semicharacter for H . Let $L_0 = L(H, \chi_0)$ denote the corresponding line bundle. Thus the decomposition $V = V_1 \oplus V_2$ distinguishes a line bundle L_0 in $\text{Pic}^H(T)$. For every line bundle $L = L(H, \chi)$ there is a point $c \in V$ uniquely determined up to translation by an element of $\Lambda(L)$, such that $L \simeq t_c^* L_0$ or equivalently $\chi = \chi_0 e^{2\pi i E(c, \cdot)}$. Here $\Lambda(L)$ is a discrete subset of V defined by

$$\Lambda(L) = \{v \in V \mid E(v, \lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\}. \quad (\text{B.9})$$

The line bundle L is called *symmetric* if $(-1)_T^* L \simeq L$ where $(-1)_T^*$ is the automorphism of the torus T coming from the map $v \mapsto -v$ on $V_{\mathbb{R}}$. It is easy to see that line bundle $L(H, \chi)$ is symmetric if and only if $\chi \subseteq \{\pm 1\}$. Thus for a symmetric line bundle the characteristic $c \in \frac{1}{2}\Lambda(L)/\Lambda(L)$.

Theta function. In this paragraph we assume that H is positive definite. Since any line bundle over V is trivial we can identify $H^0(L)$ with $H^0(\mathcal{O}_V)^\Lambda$, the subspace of holomorphic functions on V invariant under the action of Λ . Recall that a line bundle $L = L(H, \chi)$ is described in terms of canonical factor of automorphy $a_{(H, \chi)}$. Thus $H^0(L)$ can be identified with the set of holomorphic functions $\vartheta : V \rightarrow \mathbb{C}$ satisfying

$$\vartheta(v^+ + \lambda^+) = a_{(H, \chi)}(\lambda^+, v^+) \vartheta(v^+) \quad (\text{B.10})$$

for all $v \in V$ and $\lambda \in \Lambda$.

To construct a theta function it is convenient to introduce a classical factor of automorphy. This differs from the canonical factor of automorphy and depends on the choice of Lagrangian decomposition $V = V_1 \oplus V_2$.

Let us recall the discussion from section 6.2 above. A choice of Lagrangian decomposition defines two natural real subspaces inside V^+ : V_1^+ and V_2^+ respectively. Notice that each of them generates V^+ as a complex vector space. The hermitian form H restricted to $V_2^+ \times V_2^+$ defines a real symmetric form. Denote by B its \mathbb{C} -linear extension. The following properties of H and B are easy to verify

$$B|_{V^+ \times V_2^+} = H|_{V^+ \times V_2^+} \quad \text{and} \quad B|_{V_2^+ \times V^+} = H|_{V_2^+ \times V^+} - 2iE|_{V_2^+ \times V^+}. \quad (\text{B.11})$$

Note that the \mathbb{C} -bilinear form B is completely determined by the Lagrangian subspace V_2 and does not depend on a choice of Lagrangian subspace V_1 . Moreover $\text{Re}(H - B)$ is positive definite on V_1^+ . This follows since $V^+ = V_2^+ + iV_2^+$, a vector $v_1 \in V_1$ can uniquely written as $v_1 = v_2^{+'} + iv_2^{+''}$:

$$\text{Re}(H - B)(v_1, v_1) = \text{Re}(2iE(v_2^{+'}, v_1^+) - 2E(v_2^{+''}, v_1^+)) = 2E(v_1^+, v_2^{+''}) = 2E(iv_2^{+''}, v_2^{+''}) = 2H(v_2^{+''}, v_2^{+''}).$$

The bilinear form B enables us to introduce the classical factor of automorphy for $L(H, \chi)$. Define $e_{(H, \chi)} : \Lambda \times V^+ \rightarrow \mathbb{C}^*$ by

$$e_{(H, \chi)}(\lambda, v^+) := \chi(\lambda) e^{\pi(H - B)(v^+, \lambda^+) + \frac{\pi}{2}(H - B)(\lambda^+, \lambda^+)}. \quad (\text{B.12})$$

A simple calculation shows that

$$e_{(H, \chi)}(\lambda, v^+) = a_{(H, \chi)}(\lambda, v^+) \frac{e^{\frac{\pi}{2}B(v^+, v^+)}}{e^{\frac{\pi}{2}B(v^+ + \lambda^+, v^+ + \lambda^+)}}$$

Therefore the classical factor of automorphy differs from the canonical factor of automorphy by a coboundary and hence defines an equivalent line bundle.

The reason for introducing the classical factor of automorphy is that $e_{(H, \chi)}$ is invariant under a shift of lattice vector from Λ_2 , while $a_{(H, \chi)}$ is not. Thus, with the classical factor of automorphy the functional equation $\tilde{\vartheta}(v^+ + \lambda^+) = e_{(H, \chi)}(\lambda, v^+) \tilde{\vartheta}(v^+)$ can be solved using Fourier transform. The solutions of this functional equation and (B.10) are related by the cocycle $\vartheta(v^+) = e^{\frac{\pi}{2}B(v^+, v^+)} \tilde{\vartheta}(v^+)$, so

$$\vartheta^c(v^+) := e^{-\pi H(v^+, c^+) - \frac{\pi}{2}H(c^+, c^+) + \frac{\pi}{2}B(v^+ + c^+, v^+ + c^+)} \sum_{\lambda \in \Lambda_1} e^{-\frac{\pi}{2}(H - B)(\lambda^+, \lambda^+) + \pi(H - B)(v^+ + c^+, \lambda^+)}. \quad (\text{B.13a})$$

solves (B.10). Note that this expression can also be rewritten as

$$\vartheta^c(v^+) = e^{\frac{\pi}{2}B(v^+, v^+) + \frac{\pi}{2}(H - B)(c_2^+, c_1^+)} \sum_{\lambda \in \Lambda_1 - c_1} e^{-\frac{\pi}{2}(H - B)(\lambda^+, \lambda^+) + \pi(H - B)(v^+ + c_2^+, \lambda^+)}. \quad (\text{B.13b})$$

Classical theta function. Let $(X_{\mathbf{T}} = \mathbb{C}^g / \Lambda_{\mathbf{T}}, H = H_{\mathbf{T}})$ denote the principally polarized abelian variety corresponding to \mathbf{T} in the Siegel upper half space. Here $\Lambda_{\mathbf{T}} = \Lambda_1 \oplus \Lambda_2$ with $\Lambda_1 = \mathbf{T}\mathbb{Z}^g$ and $\Lambda_2 = \mathbb{Z}^g$. This decomposition induces a decomposition $\mathbb{C}^g = V_1 \oplus V_2$ where $V_1 = \mathbf{T}\mathbb{R}^g$ and $V_2 = \mathbb{R}^g$ and we can write every $v \in \mathbb{C}^g$ uniquely as $v = \mathbf{T}v^1 + v^2$.

Note that for all $v, w \in \mathbb{C}^g$ we have $B(v, w) = v^t(\text{Im } \mathbf{T})^{-1}w$ and $(H - B)(v, w) = -2iv^t w^1$. The classical *Riemann theta function* with the characteristics $\begin{bmatrix} c^1 \\ c^2 \end{bmatrix}$ is defined by

$$\vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} (v, \mathbf{T}) = \sum_{\ell \in \mathbb{Z}^g} \exp[i\pi(\ell + c^1)^t \mathbf{T}(\ell + c^1) + 2\pi i(v + c^2)^t(\ell + c^1)]. \quad (\text{B.14})$$

From eq. (B.13) it follows that the functions $\vartheta_{\mathbf{T}}^c$ and $\vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix}$ are related by

$$\vartheta_{\mathbf{T}}^{\mathbf{T}c^1 + c^2}(v) = e^{\frac{\pi}{2} B(v, v) - i\pi c^1 c^2} \vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} (v, \mathbf{T}). \quad (\text{B.15})$$

The classical Riemann theta function has the following properties:

1. For every $c^1, c^2 \in \mathbb{R}^g$ it satisfies the functional equation

$$\vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} (v + \mathbf{T}\lambda^1 + \lambda^2, \mathbf{T}) = e^{2\pi i(c^1 \lambda^2 - c^2 \lambda^1) - i\pi \lambda^1 \mathbf{T} \lambda^1 - 2\pi i v \lambda^1} \vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} (v, \mathbf{T}) \quad (\text{B.16})$$

for all $v \in \mathbb{C}^g$ and $\lambda^1, \lambda^2 \in \mathbb{Z}^g$.

2. For all $c^1, c^2 \in \mathbb{R}^g$

$$\vartheta \begin{bmatrix} c^1 + \ell^1 \\ c^2 + \ell^2 \end{bmatrix} = e^{2\pi i(\ell^2)^t c^1} \vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} \quad (\text{B.17})$$

if and only if $\ell^1, \ell^2 \in \mathbb{Z}^g$.

3. Change of characteristic translates to

$$\vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} (v, \mathbf{T}) = e^{i\pi c^1 \mathbf{T} c^1 + 2\pi i c^1 (v + c^2)} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (v + \mathbf{T}c^1 + c^2, \mathbf{T}). \quad (\text{B.18})$$

4. It satisfies the following heat equation:

$$\frac{\partial^2 \vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix}}{\partial v_I \partial v_J} (v, \mathbf{T}) = 2\pi i(1 + \delta_{IJ}) \frac{\partial \vartheta \begin{bmatrix} c^1 \\ c^2 \end{bmatrix}}{\partial \mathbf{T}_{IJ}} (v, \mathbf{T}) \quad (\text{B.19})$$

Dependence on a choice of Lagrangian decomposition. Suppose we have chosen another Lagrangian decomposition $\Lambda = \Lambda'_1 \oplus \Lambda_2$ with the same Λ_2 . Since the \mathbb{C} -bilinear form B is completely determined by V_2 that theta function is

$$\vartheta_{\Lambda'_1}^c(v) = e^{-\pi H(v, c) - \frac{\pi}{2} H(c, c) + \frac{\pi}{2} B(v + c, v + c)} \sum_{\lambda' \in \Lambda'_1} e^{-\frac{\pi}{2} (H - B)(\lambda', \lambda') + \pi (H - B)(v + c, \lambda')}. \quad (\text{B.20})$$

The Lagrangian decompositions $\Lambda'_1 \oplus \Lambda_2$ and $\Lambda_1 \oplus \Lambda_2$ are related by a linear transformation $f : \Lambda_1 \rightarrow \Lambda_2$ satisfying

$$E(f(\lambda_1), \mu_1) + E(\lambda_1, f(\mu_1)) = 0. \quad (\text{B.21})$$

Explicitly, a lattice element λ can be written in two ways $\lambda = \lambda_1 + \lambda_2$ and $\lambda = \lambda'_1 + \lambda'_2$ where

$$\lambda'_1 = \lambda_1 + f(\lambda_1) \quad \text{and} \quad \lambda'_2 = \lambda_2 - f(\lambda_1). \quad (\text{B.22})$$

Further we will denote Λ'_1 by Λ_1^f . Using this representation we express the sum over Λ_1^f in terms Λ_1

$$\vartheta_{\Lambda_1^f}^c(v) = e^{-\pi H(v,c) - \frac{\pi}{2} H(c,c) + \frac{\pi}{2} B(v+c, v+c)} \sum_{\lambda \in \Lambda_1} e^{-\frac{\pi}{2} (H-B)(\lambda, \lambda) + \pi (H-B)(v+c, \lambda)} e^{-i\pi E(f(\lambda), \lambda)}.$$

Consider the function $F(\lambda) = e^{-i\pi E(f(\lambda), \lambda)}$ defined on Λ_1 . It is easy to see that this function satisfies the equation

$$F(\lambda_1 + \mu_1) = F(\lambda_1)F(\mu_1)$$

and thus there exist an element $w_f \in \frac{1}{2}\Lambda_2$ such that

$$F(\lambda_1) = e^{2\pi i E(w_f, \lambda_1)}. \quad (\text{B.23})$$

In fact, if $\{E^j\}$ is a basis for the Lagrangian subspace V_1 and $\{F_j\}$ is a dual basis for V_2 , then the map f is represented by an integral symmetric matrix f^{jk} and $w_f = \frac{1}{2}f^{jj}F_j$. Thus we obtain that

$$\theta_{\Lambda_1^f}^c(v) = e^{i\pi E(w_f, c)} \theta_{\Lambda_1}^{c+w_f}(v). \quad (\text{B.24})$$

One sees that the change $\Lambda_1 \mapsto \Lambda_1^f$ corresponds to a change of the characteristic $c_2 \mapsto c_2 + w_f$ ¹⁵.

C Path integral measure

Let X be an n -manifold. A metric g_X on X defines a Hodge metric on the vector space of differential forms. In our conventions all forms are *dimensionless*. However the Hodge $*$ operation is dimensionfull: dimension of $*\omega_p$ is $[L]^{n-2p}$. It is convenient to introduce a dimensionless norm

$$\|\omega_p\|_L^2 := L^{2p-n} \int_X \omega_p \wedge *\omega_p \quad (\text{C.1})$$

where L is some parameter of dimension of length, $[L]$. The operator $d^\dagger|_{\Omega^p} = (-1)^{np+n+1} * d *$ has dimension $[L]^{-2}$. We also introduce the dimensionless Laplace operator

$$\Delta_p^L := L^2(dd^\dagger + d^\dagger d)|_{\Omega^p}. \quad (\text{C.2})$$

¹⁵Sometimes it is incorrectly said that it corresponds to a change of spin structure.

Gauge fields. Consider a gauge potential $a \in \Omega^{g+1}(X)$. Denote by \mathcal{G}_{g+1} the group of gauge transformations $a \mapsto a + \omega_{g+1}$ where $\omega_{g+1} \in \Omega_{\mathbb{Z}}^{g+1}(X)$. In this paragraph we want to obtain a formula for

$$\int_{\Omega^{g+1}(X)} \frac{\mathcal{D}a}{\text{Vol}(\mathcal{G}_{g+1})}$$

Using the Hodge decomposition we can write a uniquely as

$$a = a^h + a^T + d\alpha_g^T$$

Here $a^h \in \mathcal{H}^{g+1}(X)$ is a harmonic form, $a^T \in \text{im } d^\dagger \cap \Omega^{g+1}(X)$ and $\alpha_g^T \in \text{im } d^\dagger \cap \Omega^g(X)$. This implies

$$\|\delta a\|_L^2 = \|\delta a^h\|_L^2 + \|\delta a^T\|_L^2 + \langle \delta \alpha^T, (L^2 d^\dagger d) \delta \alpha^T \rangle_L \quad (\text{C.3})$$

Thus

$$\int_{\Omega^{g+1}(X)} \mathcal{D}a = \int_{\mathcal{H}^{g+1}(X)} \mathcal{D}a^h \int \mathcal{D}a^T \mathcal{D}\alpha_g^T [\det'(L^2 d^\dagger d|_{\Omega^g(X) \cap \text{im } d^\dagger})]^{1/2}. \quad (\text{C.4})$$

It is convenient to introduce the notation

$$L_p := \det'(L^2 d^\dagger d|_{\Omega^p(X) \cap \text{im } d^\dagger}). \quad (\text{C.5})$$

The gauge group \mathcal{G}_{g+1} has several connected components labelled by the harmonic forms with integral periods $\mathcal{H}_{\mathbb{Z}}^{g+1}(X)$. Using the Hodge decomposition we can write

$$\int_{\Omega^{g+1}(X)} \frac{\mathcal{D}a}{\text{Vol}(\mathcal{G}_{g+1})} = \int_{\mathcal{H}^{g+1}(X)/\mathcal{H}_{\mathbb{Z}}^{g+1}(X)} \mathcal{D}a^h \int \frac{\mathcal{D}a^T \mathcal{D}\alpha_g^T}{\text{Vol}(\mathcal{G}_{g+1}^0)} L_g^{1/2} \quad (\text{C.6})$$

where $\mathcal{G}_{g+1}^0 \cong \Omega_{\text{exact}}^{g+1}(X)/\mathcal{G}_g$ is the connected component of the identity of the gauge group \mathcal{G}_{g+1} .

Volume of the gauge algebra. To calculate the volume $\text{Vol}(\mathcal{G}_{g+1}^0)$ we notice that

$$\text{Vol}(\mathcal{G}_{g+1}^0) = \int_{\Omega_{\text{exact}}^{g+1}/\mathcal{G}_g} \mathcal{D}\alpha_{g+1} = \int_{\Omega^g} \frac{\mathcal{D}\alpha_g}{\text{Vol}(\mathcal{G}_g)} \quad (\text{C.7})$$

Using the Hodge decomposition we can write $\alpha_g = \alpha_g^h + \alpha_g^T + d\alpha_{g-1}^T$, and thus

$$\text{Vol}(\mathcal{G}_{g+1}^0) = \int_{\mathcal{H}^g(X)/\mathcal{H}_{\mathbb{Z}}^g(X)} \mathcal{D}\alpha_g^h \int \frac{\mathcal{D}\alpha_g^T \mathcal{D}\alpha_{g-1}^T}{\text{Vol}(\mathcal{G}_g^0)} L_{g-1}^{1/2}. \quad (\text{C.8})$$

The integral over the harmonic forms is a finite dimensional integral which yields the volume $V_p := \text{Vol}_L(\mathcal{H}^p/\mathcal{H}_{\mathbb{Z}}^p)$ of the harmonic torus. The volume of the harmonic torus is

$$V_p = L^{b_p(2p-n)/2} \left[\det \int_X \omega_\alpha \wedge * \omega_\beta \right]^{1/2} \quad (\text{C.9})$$

where $b_p = \dim \mathcal{H}^p$, and $\{\omega_\alpha\}$ is an *integral* basis of harmonic p -forms ($\alpha = 1, \dots, b_p$). Notice that V_p does not depend on a choice of the integral basis $\{\omega_\alpha\}$.

Successively applying the formula (C.8) one finds that all terms $\int \mathcal{D}\alpha_p^T$ but one cancel, and

$$\text{Vol}(\mathcal{G}_{g+1}^0) = \int \mathcal{D}\alpha_g^T \prod_{p=0}^g \left[V_p L_{p-1}^{1/2} \right]^{(-1)^{g-p}}. \quad (\text{C.10})$$

Combining this result with (C.6) one finds

$$\int_{\Omega^{g+1}} \frac{\mathcal{D}a}{\text{Vol}(\mathcal{G}_{g+1})} = \int_{\mathcal{H}^{g+1}/\mathcal{H}_{\mathbb{Z}}^{g+1}} \mathcal{D}a^h \int_{\Omega^{g+1} \cap \text{im } d^\dagger} \mathcal{D}a^T \left\{ \prod_{p=0}^g \left[\frac{\det'(L^2 d^\dagger d)_p}{V_p^2} \right]^{(-1)^{g-p}} \right\}^{1/2} \quad (\text{C.11})$$

D Metric variation

In this appendix we present the derivation of some well known facts about the stress-energy tensor of the self-dual field, using the trick (8.3a).

Formula for $T_{\mu\nu}^\pm$. From equation (8.4) one finds

$$\delta g^{\mu\nu} T_{\mu\nu}(F^\pm) \text{vol}(g) = \frac{\pi}{8} [F \wedge * \xi_g F - F \wedge \xi_g * F \pm F \wedge * \xi_g * F \mp F \wedge \xi_g F]. \quad (\text{D.1})$$

Now from the identity $0 = \delta_g(*(*F)) = 0$ we obtain the following equation

$$0 = -\text{tr}(\delta g^{-1} g) F + *(\xi_g * F) + \xi_g F \quad (\text{D.2})$$

expressing $\xi_g * F$ through $\xi_g F$. Thus

$$F \wedge *(\xi_g * F) = -F \wedge \xi_g F \quad \text{and} \quad F \wedge (\xi_g * F) = \text{tr}(\delta g^{-1} g) F \wedge *F - F \wedge * \xi_g F. \quad (\text{D.3})$$

Substituting these equations into (D.1) one finds

$$\delta g^{\mu\nu} T_{\mu\nu}(F^\pm) \text{vol}(g) = \frac{\pi}{4} \left[\xi_g F \wedge (*F \pm F) - \frac{1}{2} \text{tr}(\delta g^{-1} g) F \wedge *F \right] \quad (\text{D.4})$$

From this equation it follows that the stress-energy tensor for a $2\ell + 1$ -form F is the sum of the stress-energy tensor for the self-dual form F^+ and the anti-self-dual form F^- :

$$T_{\mu\nu}(F^+ + F^-) = T_{\mu\nu}(F^+) + T_{\mu\nu}(F^-). \quad (\text{D.5})$$

E Splitting of the sum over instantons

In this appendix we spell out the splitting theorems relevant to splitting the sum over instantons in a theory such as that described in section 2. The splitting is in terms of a sum over “conformal blocks.” In the present case the “conformal blocks” are theta functions of level k . The main splitting theorem is Theorem E.1 below. We then show how the failure to include subtle phases such as the quadratic refinement Ω can change the set of conformal blocks deduced from this splitting technique. This is exhibited explicitly in Theorem E.2. In several papers in the literature on M5-branes the sum over instantons is incorrectly written as the untwisted sum, rather than as the twisted sum.

E.1 Symplectic structure, complex structure, and metric

Let $V_{\mathbb{R}}$ be a real vector space with symplectic form ω which is \mathbb{Z} -valued on a lattice $\Gamma \subset V_{\mathbb{R}}$ of full rank $2g$. Now we choose a Lagrangian decomposition $V_1 \oplus V_2$ of the vector space $V_{\mathbb{R}}$ such that $\Gamma_1 \oplus \Gamma_2 = (V_1 \cap \Gamma) \oplus (V_2 \cap \Gamma)$ is the decomposition of the lattice Γ . Choose an integral basis $\{\alpha_I\}$ ($I = 1, \dots, g$) for Γ_1 and a complementary basis $\{\beta^J\}$ for Γ_2 :

$$\omega(\alpha_I, \alpha_J) = 0 = \omega(\beta^I, \beta^J) \quad \text{and} \quad \omega(\alpha_I, \beta^J) = \delta_I^J. \quad (\text{E.1})$$

We now assume there is also a complex structure J (e.g. in the geometrical setting $J = -*$ is defined by the Hodge star) which is compatible with the symplectic form ω , so we have a metric

$$g(u, v) := \omega(J \cdot u, v). \quad (\text{E.2})$$

Now choose the basis ζ_I of type $(1, 0)$. By definition ζ^I is a basis of solutions of the equation $J \cdot \zeta_I = i\zeta_I$. One can express the complex structure J in terms of the components of the complex period matrix \mathbf{T}_{IJ} . To this end we choose a basis ζ_I of the form

$$\zeta_I = \alpha_I + \mathbf{T}_{IJ}\beta^J. \quad (\text{E.3})$$

From $g(\zeta_I, \zeta_J) = g(\zeta_J, \zeta_I)$ we learn that \mathbf{T}_{IJ} is symmetric, and g is of type $(1, 1)$. Note that

$$g(\zeta_I, \bar{\zeta}_J) = 2 \operatorname{Im} \mathbf{T}_{IJ} \quad (\text{E.4})$$

which implies that $\operatorname{Im} \mathbf{T}_{IJ}$ is a positive definite matrix. If we write $\mathbf{T} = X + iY$ then the complex structure can be written as

$$J \cdot \begin{pmatrix} \alpha_I \\ \beta^I \end{pmatrix} = \begin{pmatrix} -(XY^{-1})_{IJ} & -(Y + XY^{-1}X)_{IJ} \\ (Y^{-1})^{IJ} & (Y^{-1}X)^I{}_J \end{pmatrix} \begin{pmatrix} \alpha_J \\ \beta^J \end{pmatrix}. \quad (\text{E.5})$$

Any element $v \in V_{\mathbb{R}}$ can be written as $v = v^+ + v^-$ where $J \cdot v^{\pm} = \pm i v^{\pm}$ and $(v^+)^* = v^-$. Suppose we are given $v = v_1^I \alpha_I + v_1^J \beta^J$ then

$$v^+ = \frac{1}{2}(1 - iJ)v = +\frac{1}{2i}(v^2 - v_1 \cdot \bar{\mathbf{T}}) \cdot Y^{-1} \cdot \zeta; \quad (\text{E.6a})$$

$$v^- = \frac{1}{2}(1 + iJ)v = -\frac{1}{2i}(v^2 - v_1 \cdot \mathbf{T}) \cdot Y^{-1} \cdot \bar{\zeta}. \quad (\text{E.6b})$$

This implies

$$g(v, v) = 2g(v^+, v^-) = (v^2 - v_1 \cdot \bar{\mathbf{T}}) \cdot Y^{-1} \cdot (v^2 - \mathbf{T} \cdot v_1). \quad (\text{E.7})$$

Let $\nu = n^I \alpha_I + m^I \beta_I$ then the metric (E.2) takes a form:

$$g(\nu, \nu) = \begin{pmatrix} n^I & m_I \end{pmatrix} \begin{pmatrix} (Y + XY^{-1}X)_{IJ} & -(XY^{-1})_I^J \\ -(Y^{-1}X)^I_J & (Y^{-1})^{IJ} \end{pmatrix} \begin{pmatrix} n^J \\ m_J \end{pmatrix} \quad (\text{E.8})$$

E.2 Theta function

Let us define the level $k/2$ theta function (by convention the level can be half integral) with characteristics θ_I, ϕ^I by the series

$$\Theta_{k/2, \gamma} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\mathbf{T}; a_1, a^2) := e^{i\pi k a_1 \cdot \mathbf{T} \cdot a_1 - i\pi k a_1 \cdot a^2 + i\pi k \theta \cdot \phi} \sum_{\{p_L^I\} \in \mathbb{Z}^g + \gamma + \theta} e^{i\pi k p_L \cdot \mathbf{T} \cdot p_L + 2\pi i k p_L \cdot (a^2 - \mathbf{T} \cdot a_1 - \phi)} \quad (\text{E.9})$$

where $\gamma^I \in \{0, \frac{1}{k}, \dots, \frac{k-1}{k}\}$. Here we assume that $\text{Im } \mathbf{T}_{IJ}$ is a positive definite matrix, and thus the series (E.9) converges absolutely. The characteristics ϕ_I and θ^I take values in \mathbb{R} . The theta function with *zero* characteristics is denoted by $\Theta_{k/2, \gamma}(\mathbf{T}; a_1, a^2)$. Notice that the series defining the theta function depends on a choice of Lagrangian decomposition $\Gamma_1 \oplus \Gamma_2$. More precisely, the sum goes over $\Gamma_1 + \gamma + \theta$ where $\gamma \in \frac{1}{k}\Gamma_1/\Gamma_1$, $\theta \in V_1$; the second characteristic ϕ take values in V_2 and the complex period matrix depends on a choice of both Γ_1 and Γ_2 .

The theta function (E.9) satisfies the following functional equation

$$\Theta_{k/2, \gamma} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\mathbf{T}; a_1 + \lambda_1, a^2 + \lambda^2) = \Omega_0(\lambda) e^{i\pi k \omega(\lambda, a)} \Theta_{k/2, \gamma} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\mathbf{T}; a_1, a^2) \quad (\text{E.10})$$

where

$$\Omega_0(\lambda) := e^{-i\pi k \lambda_1 \cdot \lambda^2 + 2\pi i k (\theta \cdot \lambda^2 - \phi \cdot \lambda_1)}. \quad (\text{E.11})$$

This equation means that the theta function is a section of the line bundle $\mathcal{L}^{\otimes k}$ over the torus $V_{\mathbb{R}}/\Gamma$. If $\theta \in \frac{1}{2}\Gamma_1$ and $\phi \in \frac{1}{2}\Gamma_2$ then the line bundle $\mathcal{L}^{\otimes k}$ is a symmetric line bundle. Note that different values of γ lead to different sections of the same line bundle, but different values of θ, ϕ lead to different line bundles.

Properties. The complex conjugate is

$$\overline{\Theta_{k/2,\gamma}\left[\frac{\theta}{\phi}\right](\mathbf{T}; a_1, a^2)} = \Theta_{k/2,\gamma}\left[\frac{\theta}{-\phi}\right](-\bar{\mathbf{T}}; a_1, -a^2). \quad (\text{E.12})$$

The theta function with the shifted characteristics is related to the original theta function by

$$\Theta_{k/2,\gamma}\left[\frac{\theta+m}{\phi+n}\right](\mathbf{T}; a_1, a^2) = e^{i\pi k(m \cdot n + m \cdot \theta - n \cdot \phi)} \Theta_{k/2,\gamma}\left[\frac{\theta}{\phi}\right](\mathbf{T}; a_1, a^2). \quad (\text{E.13})$$

Modular transformations. The change of symplectic basis $\{\alpha_I, \beta^I\}$ and Lagrangian decomposition is described by the group $Sp(2g, \mathbb{Z})$. It consists of the matrices of the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad D^t A - B^t C = \mathbb{1}_g, \quad D^t B = B^t D, \quad C^t A = A^t C. \quad (\text{E.14})$$

The generators can be chosen to be

1. $\begin{pmatrix} A & 0 \\ 0 & A^{-1,t} \end{pmatrix}$, $A \in GL(g, \mathbb{Z})$ i.e. $\det A = \pm 1$. This transformation describes change of integral basis in Lagrangian subspaces V_1 and V_2 .
2. $\begin{pmatrix} \mathbb{1}_g & B \\ 0 & \mathbb{1}_g \end{pmatrix}$ where B is symmetric $g \times g$ matrix. This transformation describes a change of Lagrangian subspace V_1 : $V_1 \oplus V_2 \rightarrow V'_1 \oplus V_2$.
3. $S = \begin{pmatrix} 0 & -\mathbb{1}_g \\ \mathbb{1}_g & 0 \end{pmatrix}$. This transformation exchanges the Lagrangian subspaces: $V_1 \oplus V_2 \rightarrow V_2 \oplus V_1$.

These generators act as follows on the theta function (E.9):

1. A-transform:

$$\Theta_{k/2,\gamma}\left[\frac{\theta}{\phi}\right](A\mathbf{T}A^t; a_1, a^2) = \Theta_{k/2,\gamma}\left[\frac{A^t\theta}{A^{-1}\phi}\right](\mathbf{T}; A^t a_1, A^{-1} a^2) \quad (\text{E.15a})$$

2. Generalization of the T-transform:

$$\begin{aligned} \Theta_{k/2,\gamma}\left[\frac{\theta}{\phi}\right](\mathbf{T} + B; a_1, a^2) &= e^{-\frac{i\pi}{2}k B_{II}\theta^I} e^{i\pi k B_{II}\gamma_I(\gamma_I-1) + 2\pi i k \sum_{I < J} B_{IJ}\gamma_I\gamma_J} \\ &\times \Theta_{k/2,\gamma}\left[\frac{\theta}{\phi - B\theta - \frac{1}{2}B_{II}}\right](\mathbf{T}; a_1, a^2 - Ba_1). \end{aligned} \quad (\text{E.15b})$$

3. S-transform:

$$\Theta_{k/2,\gamma}\left[\frac{\theta}{\phi}\right](\mathbf{T}^{-1}; a_1, a^2) = \det(-i\mathbf{T})^{1/2} k^{-g/2} \sum_{\gamma' \in (\frac{1}{k}\mathbb{Z}/\mathbb{Z})^{\otimes g}} e^{-2\pi i k \gamma'_I \gamma_I} \Theta_{k/2,\gamma'}\left[\frac{-\phi}{\theta}\right](\mathbf{T}; -a^2, a_1). \quad (\text{E.15c})$$

E.3 Splitting the twisted sum

We want to express

$$\mathcal{Z}_{p,q}(a) := \sum_{R \in \Gamma} \Omega_{pq}(R) e^{-\frac{\pi p}{2q} q(R-qa, R-qa) + i\pi p \omega(a, R)} \quad (\text{E.16})$$

in terms of theta functions for the complex tori $V_{\mathbb{R}}/\Gamma$. Here $a = a_1^I \alpha_I + a_I^2 \beta^I$. $\Omega_{pq}(R)$ take values in $\{\pm 1\}$ and is a quadratic refinement of $(-1)^{pq\omega}$:

$$\Omega_{pq}(R + R') = \Omega_{pq}(R) \Omega_{pq}(R') e^{i\pi qp \omega(R, R')}. \quad (\text{E.17})$$

The results of this subsection can be summarized follows:

Lemma E.1. *If $pq = p \pmod{2}$ then the twisted sum $\mathcal{Z}_{p,q}(a)$ defines a section of a line bundle $\mathcal{L}^{\otimes pq}$ over the finite dimensional tori $V_{\mathbb{R}}/\Gamma$.*

Theorem E.1. *If $pq = p \pmod{2}$ and $\gcd(p, q) = 1$ then the twisted sum $\mathcal{Z}_{p,q}(a)$ splits*

$$\mathcal{Z}_{p,q}(a) = (\det \frac{2g}{p} \text{Im } \mathbf{T})^{1/2} \sum_{\substack{\gamma_p \in (\frac{1}{p}\mathbb{Z}/\mathbb{Z})^g \\ \gamma_q \in (\frac{1}{q}\mathbb{Z}/\mathbb{Z})^g}} \Theta_{pq/2, \gamma_p + \gamma_q} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\mathbf{T}; 0, 0) \overline{\Theta_{pq/2, \gamma_p - \gamma_q} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\mathbf{T}; a_1, a^2)} \quad (\text{E.18})$$

where $\Theta_{pq/2, \gamma} \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (\mathbf{T}; a_1, a^2)$ is a theta function with characteristics defined by series (E.9), $a = a_1^I \alpha_I + a_I^2 \beta^I$.

In particular, if $p = q = 1$ then the twisted sum (E.16) factorizes as the square of a *single* theta function.

Remark E.1. *If $\gcd(p, q) = m \neq 1$ then the twisted sum also splits, however it splits in terms of level pq/m^2 theta functions on the torus $V_{\mathbb{R}}/mV_{\mathbb{Z}}$.*

Proof of the lemma. The function $\psi(a)$ descends to a section of a line bundle \mathcal{L}^{pq} if satisfies the equation

$$\psi(a + \lambda) = \Omega_{pq}(\lambda) e^{i\pi pq \omega(a, \lambda)} \psi(a). \quad (\text{E.19})$$

It is matter of simple algebra to show that the sum (E.16) satisfies equation (E.19) iff $pq = p \pmod{2}$, where we use the fact that $\Omega_{pq}(q\lambda) = \Omega_{pq}(\lambda)$. \square

Proof of the theorem. We write $R = n^I \alpha_I + m_I \beta^I$. Then we can write the quadratic refinement Ω_{pq} in the form

$$\Omega_k(R) = e^{-i\pi kn \cdot m + 2\pi i k(m \cdot \theta - n \cdot \phi)}. \quad (\text{E.20})$$

Here θ and ϕ are half integrally quantized, so that $\Omega_k(R) \in \{\pm 1\}$.

To split the sum we need to do Poisson resummation over m_I . The result is:

$$\begin{aligned} \mathcal{Z}_{p,q}(a) &= (\det \frac{2q}{p} \operatorname{Im} \mathbf{T})^{1/2} e^{-i\pi k a_1 \cdot \bar{\mathbf{T}} \cdot a_1 + i\pi k a_1 \cdot a^2} \sum_{n^I, w^I} \exp \left[i\pi k p_L \cdot \mathbf{T} \cdot p_L - i\pi k p_R \cdot \bar{\mathbf{T}} \cdot p_R \right] \\ &\quad \times \exp \left[-2\pi i k p_R \cdot (a^2 - \bar{T} \cdot a_1) - 2\pi i k q \phi \cdot (p_L + p_R) \right] \end{aligned} \quad (\text{E.21})$$

where

$$p_L^I = \frac{1}{2q} n^I + \frac{1}{2} \left[qn^I + 2q\theta^I + \frac{2}{p} w^I \right] \quad \text{and} \quad p_R^I = \frac{1}{2q} n^I - \frac{1}{2} \left[qn^I + 2q\theta^I + \frac{2}{p} w^I \right]. \quad (\text{E.22})$$

From the conditions $pq = p \pmod{2}$ and $\gcd(p, q) = 1$ it follows that for $q = \text{odd}$, p an arbitrary integer s.t. $\gcd(p, q) = 1$, we can write $q = 2r + 1$ and shift $w^I \mapsto w^I - prn^I - pr2\theta^I$:

$$p_L^I = \frac{1}{2q} n^I + \frac{1}{2} \left[n^I + 2\theta^I + \frac{2}{p} w^I \right] \quad \text{and} \quad p_R^I = \frac{1}{2q} n^I - \frac{1}{q} \left[qn^I + 2q\theta^I + \frac{2}{p} w^I \right]. \quad (\text{E.23})$$

Now introduce

$$n^I = qt^I + 2q\gamma_q^I \quad \text{and} \quad w^I = \frac{1-q}{2} pt^I - pq\gamma_q^I + ps^I + p\gamma_p^I.$$

where $s^I, t^I \in \mathbb{Z}$, $\gamma_q^I \in \{0, \frac{1}{q}, \dots, \frac{q-1}{q}\}$ and $\gamma_p^I \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$. Thus

$$p_L^I = \underbrace{t^I + s^I}_{n_L} + \gamma_q^I + \gamma_p^I + \theta^I \quad \text{and} \quad p_R^I = \underbrace{-s^I}_{n_R} + \gamma_q^I - \gamma_p^I + \theta^I. \quad (\text{E.24})$$

One sees that n_L and n_R are independent summation variables. Thus the sum (E.21) splits as in (E.18). \square

E.4 Splitting the untwisted sum

We want to express

$$\mathcal{Z}_{p,q}(a) := \sum_{R \in V_{\mathbb{Z}}} e^{-\frac{\pi p}{2q} g(R - qa, R - qa) + i\pi p \omega(a, R)} \quad (\text{E.25})$$

in terms of theta functions for the complex tori $V_{\mathbb{R}}/V_{\mathbb{Z}}$.

Theorem E.2. *If $pq = p \pmod{2}$ then the untwisted sum (E.25) defines a section of a vector bundle (not a line bundle) over $V_{\mathbb{R}}/V_{\mathbb{Z}}$. If in addition $\gcd(p, q) = 1$ then the untwisted sum $\mathcal{Z}_{p,q}(a)$ splits:*

(a) *if $p = \text{odd}$ and $q = \text{odd}$*

$$\mathcal{Z}_{p,q}(a) = (\det \frac{2q}{p} \operatorname{Im} \mathbf{T})^{1/2} \frac{1}{2^g} \sum_{\theta, \phi \in \{0, \frac{1}{2}\}^g} \sum_{\substack{\gamma_p \in (\frac{1}{p}\mathbb{Z}/\mathbb{Z})^g \\ \gamma_q \in (\frac{1}{q}\mathbb{Z}/\mathbb{Z})^g}} \Theta_{pq/2, \gamma_p + \gamma_q} \left[\frac{\theta}{\phi} \right] (\mathbf{T}; 0, 0) \overline{\Theta_{pq/2, \gamma_p - \gamma_q} \left[\frac{\theta}{\phi} \right] (\mathbf{T}; a_1, a^2)} \quad (\text{E.26a})$$

(b) if $p = \text{even}$ and $q = \text{odd}$

$$\mathcal{Z}_{p,q}(a) = (\det \frac{2q}{p} \text{Im } \mathbf{T})^{1/2} \sum_{\substack{\gamma_p \in (\frac{1}{p}\mathbb{Z}/\mathbb{Z})^g \\ \gamma_q \in (\frac{1}{q}\mathbb{Z}/\mathbb{Z})^g}} \Theta_{pq/2, \gamma_p + \gamma_q}(\mathbf{T}; 0, 0) \overline{\Theta_{pq/2, \gamma_p - \gamma_q}(\mathbf{T}; a_1, a^2)} \quad (\text{E.26b})$$

Here $\Theta_{pq/2, \gamma}[\frac{\theta}{\phi}](\mathbf{T}; a_1, a^2)$ is the theta function with characteristics defined by the series (E.9), and $a = a_1^I \alpha_I + a_I^2 \beta^I$.

Remark E.2. If $\gcd(p, q) = m \neq 1$ then the untwisted sum also splits, however it splits in terms of the level pq/m^2 theta functions on the torus $V_{\mathbb{R}}/mV_{\mathbb{Z}}$.

Proof of the theorem. Decompose $a = a_1^I \alpha_I + a_I^2 \beta^I$ and $R = n^I \alpha_I + m_I \beta^I$. To split the sum (E.25) we need to do Poisson resummation over m_I :

$$\mathcal{Z}_{p,q}(a) = (\det \frac{2q}{p} Y)^{1/2} e^{-i\pi k a_1 \cdot \bar{\mathbf{T}} \cdot a_1 + i\pi k a_1 \cdot a^2} \sum_{n^I, w^I} \exp \left[i\pi k p_L \cdot \mathbf{T} \cdot p_L - i\pi k p_R \cdot \bar{\mathbf{T}} \cdot p_R + 2\pi i k p_R \cdot (\bar{\mathbf{T}} \cdot a_1 - a^2) \right] \quad (\text{E.27})$$

where $Y = \text{Im } \mathbf{T}$, $k = qp$ and

$$p_L^I = \frac{1}{2q} n^I + \frac{1}{p} w^I \quad \text{and} \quad p_R^I = \frac{1}{2q} n^I - \frac{1}{p} w^I. \quad (\text{E.28})$$

Now we have to consider two cases: ($p = \text{odd}, q = \text{odd}$) and ($p = \text{even}, q = \text{odd}$) separately.

q = odd, p = odd: We split n^I and w^I in (E.28) as follows

$$w^I = p t^I + p \gamma_p^I \quad \text{and} \quad n^I = 2q s^I + 2q \gamma_q^I + 2q \theta^I$$

where $s^I, t^I \in \mathbb{Z}$, $\theta^I \in \{0, \frac{1}{2}\}$, $\gamma_p^I \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ and $\gamma_q^I \in \{0, \frac{1}{q}, \dots, \frac{q-1}{q}\}$. Thus we have

$$p_L^I = \underbrace{s^I + t^I}_{n_L} + \theta^I + \gamma_q^I + \gamma_p^I \quad \text{and} \quad p_R^I = \underbrace{s^I - t^I}_{n_R} + \theta^I + \gamma_q^I - \gamma_p^I.$$

We certainly want to consider $n_L = s + t$ and $n_R = s - t$ as independent summation variables however they are not independent: $n_L^I - n_R^I$ are even integers for all I . This difficulty can be overcome by inserting the following function into the sum (E.27):

$$\frac{1}{2^g} \sum_{\phi_I \in \{0, \frac{1}{2}\}} e^{2\pi i k \phi_I (n_R - n_L)^I} = \begin{cases} 1, & \text{iff the integers } (n_L - n_R)^I \text{ are even for all } I; \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $k = qp = \text{odd}$ and $\gcd(p, q) = 1$ the sum (E.27) can be written in term of the level k theta functions with characteristics as in (E.26a).

q = odd, p = even: Now we do the following change of variable in (E.28): $n^I = qt^I + 2q\gamma_q^I$ and $w^I = (p/2)t^I + ps^I + p\gamma_p^I$ where $t^I \in \mathbb{Z}$. In this case equations (E.28) take the form

$$p_L^I = \underbrace{t^I + s^I}_{n_L} + \gamma_p^I + \gamma_q^I \quad \text{and} \quad p_R^I = \underbrace{-s^I}_{n_R} + \gamma_p^I - \gamma_q^I.$$

The variables n_L and n_R are independent summation variables, and thus we obtain (E.26b). \square

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